T O O L S

A Mathematical Sketch and Model Book

BY

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## PREFACE

Somewhat in the nature of an experiment, this book has been designed especially for college students who are prospective teachers of mathematics. It serves not only to focus their attention upon the geometrical tool and the precise manner in which it is used, but also furnishes them with abundant material that can and should be introduced into high school work.

The subject matter presented here requires no preliminary knowledge of mathematics in advance of that acquired in the standard freshman courses of Algebra, Trigonometry, and Analytics. Few difficulties will be encountered even if the book is studied at the freshman level.

Since there are already a number of excellent available texts dealing with modern geometry, this subject has been sacrificed to a large extent to make room for material believed to be more adaptable to the needs of the prospective teacher.

The arrangement is based upon the three-hour-per-week class. It is suggested that two of these hours be spent in the classroom, the third in the laboratory. Thus, at the average rate of two plates per week, the material will be found ample for a year course. Since, generally, any section is independent of the others, the course can be arranged to meet the desires of the group. Furthermore, a student entering the course at the beginning of the second semester will not necessarily be handicapped if the order of the book is followed.

There are approximately 80 plates, each faced by explanatory text and each designed as a class-hour unit. Sufficient space is provided for answers to questions.

The full value of the book can be realized only by some thought and much labor. The student should make free use of color in completing the drawings. The essential role of some vital parts of a complicated configuration is more clearly presented if they appear in color. A supplementary notebook with ring binder will be found useful in keeping models and notes that cannot be inserted herein.

Much depends upon the instructor. It should be clear that there is no attempt to encourage mechanical perfection on the part of the student in the art of drafting. Instead, it is hoped that this will bring a more thorough and sympathetic understanding of geometrical structure. In completing drawings and making suggested models, it is hoped that the student will develop the feeling of being co-author. In the end, he will have a volume containing a record of his own creative efforts, a volume that may serve him later as a source of supplementary material in his career as a teacher.

The equipment for the laboratory is inexpensive. The following should be included:

Thin colored art paper (standard size pads).
Thin tracing paper having a wax body or finish.
Straightedge, Compasses, and Dividers.
Colored poster-type cardboard about 12 ply.
Eyelet punch.
Eyelets, #2 and #3.
Phototrimmer - medium or large.

Although the material of this book was gathered from many sources, the following were of special service throughout:

Adler, A. : Theorie der geometrischen Konstruktionen, Leipzig (1906) (Out of print).

Fourrey, E. : Procédés originaux de Constructions géométriques, Paris (1924) (At present unobtainable).

Hudson, H. P. : Ruler & Compasses, London, (1916).

Kempe, A. B. : How to Draw a Straight Line, New York (1877) (Out of print and rare).

Row, T. S. : Geometrical Exercises in Paper Folding, Madras (1893) (translated by Beman and Smith, Chicago, 1901) (Out of print).

The author wishes to thank Professor E. H. C. Hildebrandt for many suggestions, Dorothy Blanchard for compiling the index and reading proof, and George Guttner for his courteous cooperation in the matter of publication.

Baton Rouge, Louisiana
June, 1941

# CONTENTS

## INTRODUCTION

Salient features and important conclusions are listed here in order that the student might gain through this broad view a general understanding of the concepts discussed herein.

Plane Euclidean constructions are those which may be effected by straightedge and compasses. Sometimes simple, sometimes complicated, they are all, nevertheless, composed of straight lines and circles. The ultimate object of any such construction is the location of points which are found as the intersection of two lines, a line and a circle, or two circles. Accordingly, any tool (such as the Angle Ruler) is equivalent to the straightedge and compasses if it is capable of making these three fundamental constructions.

Plane Euclidean constructions have for their algebraic interpretations equations whose roots are at most quadratic irrationalities. For the most part, such equations are of degree not higher than the second. Tools (or systems*) which will produce such constructions are thus called quadratic. Into this classification fall the unassisted Compasses, the Parallel Ruler, the Marked Ruler, the system of Straightedge and Fixed Circle, etc. Those tools which will effect constructions equivalent to equations of degree as high as the fourth are called quartic. These include the Marked Ruler, the Compasses and Fixed Conic, the Carpenter's Square, the Tomahawk, etc.

The importance of the discussion of Cubics and Quartics preliminary to the analysis of Higher Tools cannot be overemphasized. It is shown that any quartic construction is reducible by means of straightedge and compasses either to the trisection of a particular angle or to the cube root of a certain segment length. The two ancient problems of Trisection and Duplication of a Cube thus appear in roles of fresh importance.

Plane linkages (compound compasses) are very complex tools. Their appearance in the midst of elementary tools is excused by an anticipation of normal curiosity. Having just completed a section devoted to straightedge constructions, it is only natural to speculate upon the existence of such an instrument. To say we build one straightedge upon another as a guide is to beg the question. A mechanical construction of a straight line or straightedge comes naturally only through the medium of plane jointed links in the manner of Peaucellier, Hart, and Kempe. In view of the fact that the simplest linkages producing line motion are those involving five bars, the time-honored straightedge seems disturbingly complex.

Two unusual designations appearing frequently throughout the book are (1) the use of the contraction "hypo" (for hypothetical) to indicate a locus which is defined but may not be drawn; and (2) the notation A(B) to indicate the circle with center A and radius AB.

The author does not wish to overburden the student by insisting upon the faithful adherence to any particular tool. For example, the location of the intersections of two hypocircles by the Parallel Ruler requires that certain perpendiculars be established. Having already erected perpendiculars in a preliminary figure, the student may conscientiously exchange the Parallel Ruler for a more adaptable tool. Such practice, moreover, would avoid many minor constructional elements that might obscure main issues and objectives.

This book is presented with the sincere hope that from it a wealth of pleasure and satisfaction may be derived. Intellectual profit will then accumulate without apparent effort.

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* Such as those discussed in Section VII.

## SECTION I

## THE STRAIGHTEDGE AND MODERN COMPASSES

### (Modern Geometry)

The Straightedge is an instrument used to establish the straight line passing through two given distinct points.

The instrument called the Modern Compasses is used to draw the circle with given center and given radius. If the radius is not given "in position" - that is, with an extremity at the center - we postulate the ability to "carry" this radius by the compasses into position. This is, in effect, absorbing the principle of the Dividers into the compasses.

These two tools and the rather restricted uses to which they are put seem scanty equipment indeed to erect any sort of geometrical structure worth the effort. This makes all the more surprising the fact that the production is intricate, elaborate, and certainly most valuable.

However elaborate a construction may be, it is but the location of points found as the intersection of

1. two lines;
2. a line and a circle;
3. two circles.

### BIBLIOGRAPHY AND FURTHER READING

Altshiller-Court, N : College Geometry, Richmond (1923).
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Casey, John : A Sequel to Euclid, Dublin (1881, 1888).
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The following pertain particularly to regular polygons.

Adler, A. : Theorie der geometrischen Konstruktionen, Leipzig (1906) 200-229.
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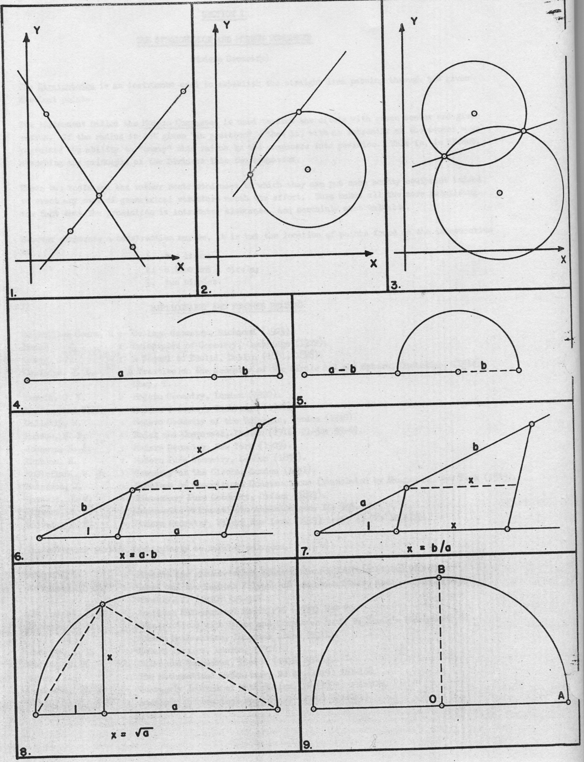
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All constructions of plane euclidean geometry are but the location of points either as the intersection of two lines, or a line and a circle, or two circles. WE SHALL PROVE THAT, UNDER THE TWO RULES GOVERNING THE USE OF THE STRAIGHTEDGE AND COMPASSES, THESE CONSTRUCTIONS CONSIST ONLY OF THE RATIONAL OPERATIONS OF ADDITION, SUBTRACTION, DIVISION, AND MULTIPLICATION TOGETHER WITH THE IRRATIONAL OPERATION OF THE EXTRACTION OF A SQUARE ROOT OF POSSESSED GEOMETRIC LENGTHS.

We shall first show that these five operations are the only possible ones in straightedge and compasses constructions. The proof is concerned with these cases:

CASE I: Fig. 1. When given four points determining two lines we may draw these lines and thereby determine their intersections with two other arbitrarily chosen perpendicular lines used as <u>reference axes</u>. With these <u>intercepts</u> known, the equations of the lines are:

$$x/a_1 + y/b_1 = 1 \qquad \text{and} \qquad x/a_2 + y/b_2 = 1$$

where the a's and b's are constructible lengths. The coordinates of their intersection point are the simultaneous solutions:

$$x = a_1a_2(b_2 - b_1)/(a_1b_2 - a_2b_1), \qquad y = b_1b_2(a_1 - a_2)/(a_1b_2 - a_2b_1).$$

Each fraction here represents a series of constructions possible by the methods shown in Figures 4, 5, 6, 7. Therefore, <u>all line constructions lead to nothing more than the rational operations of addition, subtraction, multiplication, and division of lengths</u>.

CASE II: Fig. 2. A given line and a given circle have for equations:

$$x/a + y/b = 1 \qquad \text{and} \qquad (x - h)^2 + (y - k)^2 = r^2.$$

To find their intersections, eliminate first x, then y, obtaining

$$Ax^2 + Bx + C = 0, \qquad Ly^2 + My + N = 0,$$

where the coefficients are constructible by Figures 4, 5, 6, 7. <u>The solutions</u>, $x = [-B \pm \sqrt{(B^2 - 4AC)}]/2A$ for instance, of the most general of these quadratics <u>involves, in addition to the rational operations, the irrational operation of extraction of square roots, but nothing further</u>.

CASE III: Fig. 3. The intersection of two given circles is the same as the intersection of their common chord and one of the circles. Thus, since the coefficients in the equation of the chord are rational functions of those in the equation of the circles, <u>this case</u> reduces immediately to II, and <u>introduces no new operation</u>.

We shall now show that these five operations are possible by straightedge and compasses and give the constructions.

Figs. 4, 5 indicate the obvious means of <u>addition</u> and <u>subtraction</u> of possessed lengths.

Figs. 6, 7 give methods of <u>multiplying</u> and <u>dividing</u> the lengths <u>a</u> and <u>b</u>. The construction in either case is that of similar triangles, involving the construction of parallel lines.

Fig. 8. This exhibits the construction for the <u>square root</u> of a length <u>a</u>. Describe the circle on (1 + a) as diameter and erect the perpendicular at the junction point. The length <u>x</u> intercepted by the arc is $\sqrt{a}$. Compare similar triangles:

Fig. 9. Locate M, the midpoint of OA = 1. Draw OB perpendicular to OA. With M as center and MB as radius describe an arc cutting AO extended at C. Calculate the lengths:

OC = BC =

## FUNDAMENTAL THEOREMS

The following theorems correspond to those selected by the National Committee on Mathematical Requirements as of greatest importance and listed by them as fundamental. These are given to the student in order for him to bridge the gap more easily between high school geometry and the materials of this course. Locate the following theorems in a Standard text and list the page references opposite each.

1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.

2. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the two triangles are congruent.

3. If two triangles have three sides of one equal respectively to three sides of the other the two triangles are congruent.

4. If two right triangles have the hypotenuse and a leg of one equal respectively to the hypotenuse and a leg of the other, they are congruent.

5. If two sides of a triangle are equal, the angles opposite these sides are equal.

6. The locus of points equally distant from two given points is the perpendicular bisector of the line joining them.

7. The locus of points equally distant from the sides of an angle is the bisector of the angle.

8. If two parallel lines are cut by a transversal, the alternate interior angles are equal.

9. If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.

10. The sum of the angles of a triangle is a straight angle. ($180^\circ$).

11. A parallelogram is divided into two congruent triangles by a diagonal.

12. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

13. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

14. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on every transversal.

15. The area of a parallelogram is equal to the product of the base and altitude.

16. The area of a triangle is equal to one half the product of its base and altitude.

17. The area of a trapezoid is equal to one half the product of its altitude and the sum of its bases.

18. The area of a regular polygon is equal to one-half the product of its apothem and its perimeter.

19. If a straight line intersects two sides of a triangle and is parallel to the third side, it divides the two sides proportionately.

20. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

21. The segments cut off on two transversals by three or more parallel lines are proportional.

22. Two triangles are similar if they have two angles of one equal respectively to two angles of the other.

FUNDAMENTAL THEOREMS

23. Two triangles are similar if an angle of one is equal to an angle of the other and the including sides are proportional.

24. Two triangles are similar if their corresponding sides are proportional.

25. If two chords intersect in a circle, the product of the parts of one is equal to the product of the parts of the other.

26. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

27. If two polygons can be divided into two triangles which are similar and similarly placed, the polygons are similar.

28. If two polygons are similar, they can be divided into triangles which are similar and similarly placed.

29. The bisector of an angle of a triangle divides the opposite side into parts proportional to the adjacent sides.

30. The areas of two similar triangles are to each other as the squares of any two corresponding sides.

31. The areas of two similar polygons are to each other as the squares of any two corresponding sides.

32. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle and to each other.

33. In any right triangle the square on the hypotenuse equals the sum of the squares of the other two sides.

34. In the same circle or in equal circles, equal central angles have equal arcs.

35. In the same circle or in equal circles, equal arcs have equal central angles.

36. In the same circle or in equal circles, two central angles are proportional to their arcs.

37. In the same circle or in equal circles, equal chords have equal arcs.

38. In the same circle or in equal circles, if two arcs are equal their chords are equal.

39. A diameter perpendicular to a chord bisects the chord and its arc.

40. A diameter which bisects a chord (not a diameter) is perpendicular to it.

41. A tangent to a circle at a given point is perpendicular to the radius drawn to that point.

42. A line perpendicular to a radius at its outer extremity is tangent to the circle.

43. In the same circle or in equal circles, equal chords are equally distant from the center.

44. In the same circle or in equal circles, chords which are equally distant from the center are equal.

45. An inscribed angle is measured by one-half its arc.

46. Angles inscribed in the same segment are equal.

47. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon, and tangents at the points of division form a regular circumscribed polygon.

48. The area of a circle is equal to one half the product of its radius and its circumference.

49. The circumference of a circle is equal to the product of its diameter and pi.

# FUNDAMENTAL CONSTRUCTIONS

The following is a list selected by the National Committee on Mathematical Requirements as fundamental constructions. Make these twenty constructions, using the straightedge and compasses as indicated at the beginning of this section. Place whatever explanatory notes are necessary in the space provided between questions.

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FIG. 1. Bisect the line segment and draw its perpendicular bisector.

FIG. 2. Bisect the given angle.

FIG. 3. Construct the perpendicular to the given line through the given point.

FIG. 4. Construct an angle at P equal to the given angle.

FIG. 5. Draw the line parallel to the given line through the given point.

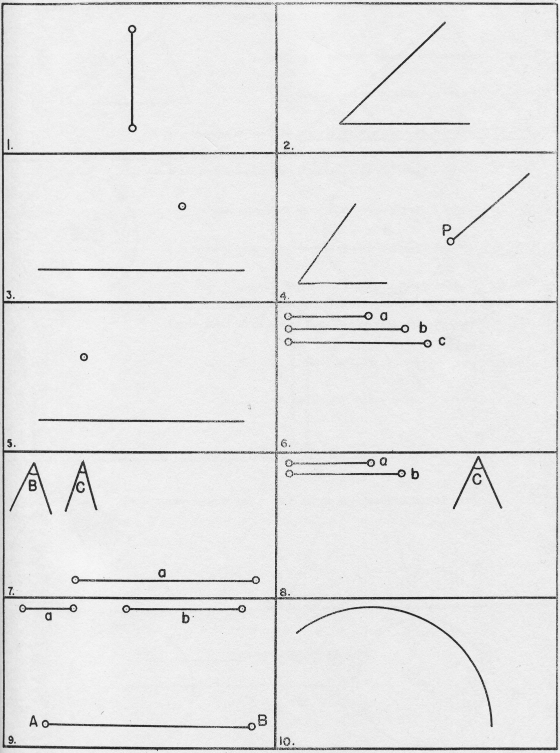
FIG. 6. Construct the triangle whose sides are the given segments, a, b, c.

FIG. 7. Construct a triangle, given two angles and the included side.

FIG. 8. Construct a triangle, given two sides, a, b, and the included angle.

FIG. 9. Divide the segment AB into parts proportional to the segments a, b.

FIG.10. Given the arc of a circle, find its center.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## FUNDAMENTAL CONSTRUCTIONS

FIG. 11. Circumscribe a circle about the given triangle.

FIG. 12. Inscribe a circle in the given triangle.

FIG. 13. Construct the tangents to the given circle from the external point P.

FIG. 14. Construct the tangent to the circle through the point P on the circle.

FIG. 15. Construct a <u>fourth</u> proportional to the three given segments a, b, c.

FIG. 16. Construct a <u>mean</u> proportional between the two given segments a, b.

FIG. 17. Construct a polygon similiar to the given polygon.

FIG. 18. Construct a triangle with area equal to that of the given polygon.

FIG. 19. Inscribe a square in the given circle.

FIG. 20. Inscribe a regular hexagon in the given circle.

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GIVE A LIST OF REFERENCE PAGES FROM A STANDARD TEXT ON THESE CONSTRUCTIONS:

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| 1. | 2. | 3. | 4. | 5. |
| 6. | 7. | 8. | 9. | 10. |
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TITLE AND DATE OF REFERENCE:____________________

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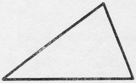

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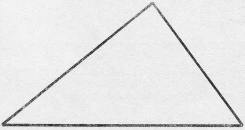

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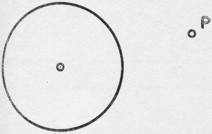


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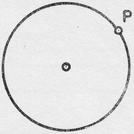


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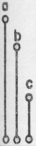


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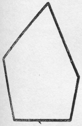

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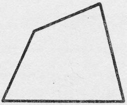

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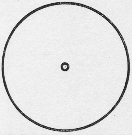

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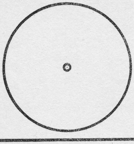

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THEOREMS OF MENELAUS & CEVA

Of prime importance to much that will follow throughout the book are the theorems of Menelaus and Ceva.

FIG. 1. The Theorem of Menelaus: ANY LINE CUTS THE SIDES (prolonged if necessary) OF A TRIANGLE SO THAT THE PRODUCT OF THREE NON-ADJACENT SEGMENTS INTO WHICH THE SIDES ARE SEPARATED EQUALS THE PRODUCT OF THE OTHER THREE NON-ADJACENT SEGMENTS.‡

Dropping perpendiculars from the vertices of the triangle to the intersecting line, we have from similar triangles:

$RA/RB = x/y$ $\qquad$ $QC/QA = z/x$ $\qquad$ $PB/PC = y/z$.

Multiplying,

$$(AR)(QC)(PB) = (BR)(AQ)(PC).^{*} \qquad \text{Q.E.D.}$$

State and prove the converse. (See Johnson, p. 146)

Fig. 2. The Theorem of Ceva: IF LINES ARE DRAWN FROM THE VERTICES OF A GIVEN TRIANGLE TO AN ARBITRARY POINT O, THEN THE PRODUCT OF THREE NON-ADJACENT SEGMENTS INTO WHICH THE SIDES ARE SEPARATED IS EQUAL TO THE PRODUCT OF THE REMAINING NON-ADJACENT SEGMENTS. THAT IS:

$$(AR)(BP)(CQ) = -(BR)(CP)(AQ).$$

In order to prove this, draw line XAY parallel to BC meeting COR and BOQ in X and Y, respectively. The similar triangles thus formed give the following proportions:

$PB/PC = AY/AX$; $\qquad$ $QC/QA = BC/AY$; $\qquad$ $RA/RB = AX/BC$.

Multiplying these together establishes the theorem. State and prove the converse.

As applications of these theorems or their converses, prove:

FIG. 3. The medians of a triangle meet in a point (the Centroid).

FIG. 4. The altitudes of a triangle meet in a point (the Orthocenter).

FIG. 5. The exterior angle bisectors meet the opposite sides of a triangle in three collinear points. (In connection, see Orthic Triangle, Plate 8,3).

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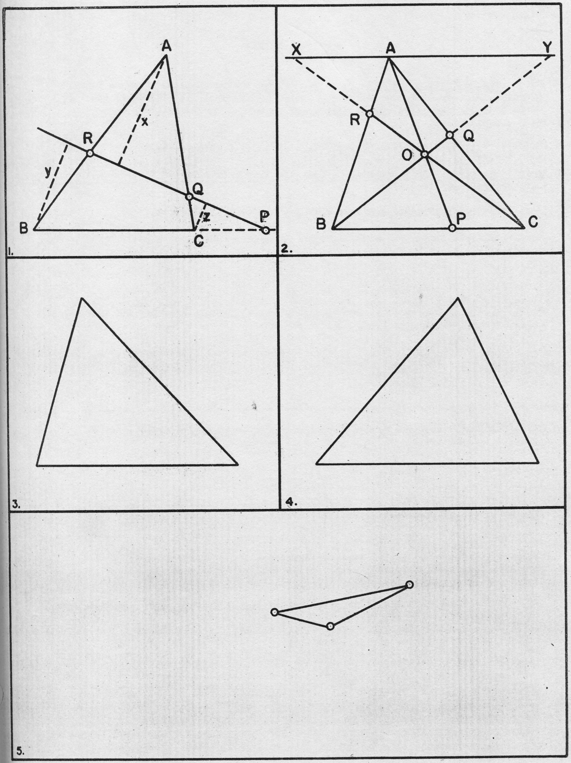
‡ Adjacent segments are those which terminate in the same vertex.

* As is customary, we agree to call the ratio PB/PC negative if P lies between B and C.

Plate 4.



## SIMILITUDE OF CIRCLES

FIG. 1. In the two given circles, $O_1(r_1)$ and $O_2(r_2)$,* we draw parallel diameters. The lines joining the extremities of these diameters meet the line of centers in the points I and E. These points are the internal and external centers of similitude of the two circles. Let the distance $O_1O_2 = k$. Now by similar triangles,

$$(O_1I)/r_1 = (O_2I)/r_2 = (O_1I + O_2I)/(r_1 + r_2) = k/(r_1 + r_2) = \text{constant}.$$

Thus $O_1I$ is a constant and I is accordingly a fixed point which is independent of the position of the constructed diameters. Furthermore,

$$(O_1E)/r_1 = (O_2E)/r_2 = (O_1E - O_2E)/(r_1 - r_2) = k/(r_1 - r_2) = \text{constant}.$$

Thus E is likewise a fixed point. Notice that these centers of similitude are the intersection of common tangents. Discuss the case when $r_1 = r_2$.

FIG. 2. Draw line segments from P to the given circle. What is the locus of the midpoints of these segments? (Hint: Compare similar triangles).

FIG. 3. Show that lines joining P, a point of intersection of two circles, to I and E bisect the angles at P which are formed by the lines joining P and the centers.

FIG. 4. Construct the three external centers of similitude of the three given circles. Show that these three points lie on a straight line. (Hint: Use the theorem of Menelaus). Notice that any pair of incenters of similitude is collinear with the other excenter of similitude.

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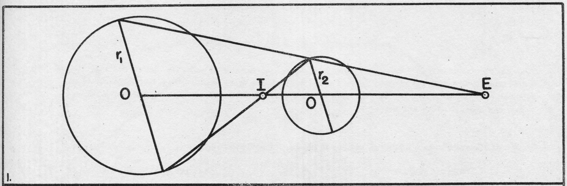
* The notation $O(r)$ signifies the circle with center O and radius r.



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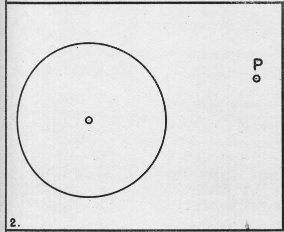

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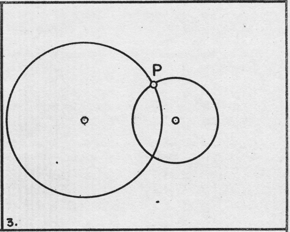

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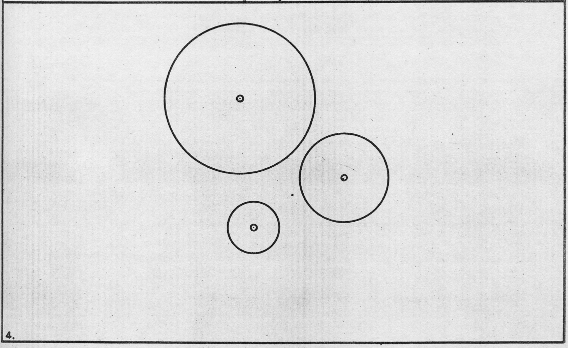
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# POWER OF A POINT AND RADICAL AXIS

FIG. 1. We now establish a very important and fundamental theorem of geometry, hereafter described as the Secant Property of the Circle. From a point P lines are drawn to intersect the given circle. Since the arc subtended by $\angle$ ACD plus that subtended by $\angle$ ABD is the entire circumference, these angles are supplementary and thus

$$\angle ACD = \angle PBD.$$

Triangles PCA and PBD are therefore similar with the proportion:

$$PB/PC = PD/PA \qquad \text{or} \qquad (PB)(PA) = (PD)(PC).$$

Thus, IF LINES ARE DRAWN FROM A FIXED POINT TO INTERSECT A FIXED CIRCLE, THE PRODUCT OF THE DISTANCES FROM THE FIXED POINT TO THE POINTS OF INTERSECTION OF EACH LINE AND CIRCLE IS CONSTANT.

FIG. 2. The constant is easily evaluated by drawing the line through P and the center of the circle. We have:

$$(PO - r)(PO + r) = p, \text{ a constant,} \qquad \text{or} \qquad (PO)^2 - r^2 = p.$$

The quantity p is called the <u>power of the point</u> P with respect to the fixed circle. If the point P is <u>outside</u>, <u>on</u>, or <u>inside</u> the circle the corresponding power is <u>positive</u>, <u>zero</u>, or <u>negative</u> respectively.

FIG. 3. Let us look for the locus of all points P that have equal power with respect to two circles, $O_1(r_1)$, $O_2(r_2)$. If P is any such point, let PM be dropped perpendicular to the line of centers. Then

$$(O_1P)^2 - r_1^2 = (O_2P)^2 - r_2^2 \qquad \text{or} \qquad (PM)^2 + (O_1M)^2 - r_1^2 = (PM)^2 + (O_2M)^2 - r_2^2.$$

Thus

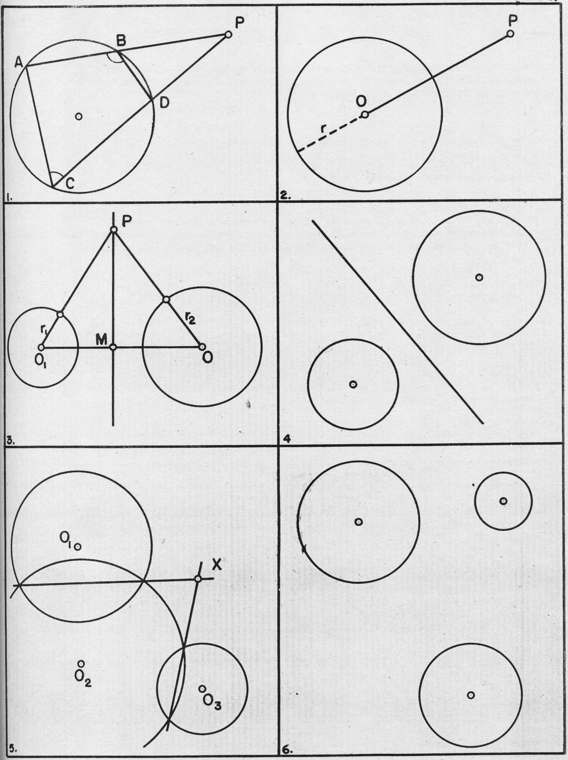
$$(O_1M)^2 - (O_2M)^2 = (O_1M - O_2M)(O_1M + O_2M) = r_1^2 - r_2^2.$$

But since $(O_1M + O_2M)$ and $r_1^2 - r_2^2$ are constants, then $(O_1M - O_2M)$ must therefore be a constant. If two quantities have their sum and difference both constants they are themselves constants. Accordingly, $O_1M$ is constant and thus M is a fixed point for any position of P. The locus of P therefore is a straight line perpendicular to the line of centers, $O_1O_2$. It is called the <u>Radical Axis</u> of the two circles.

FIG. 4. Show that for all points on the radical axis, the tangent lengths drawn to the circles are equal. Notice that if the circles intersect, the Radical Axis is their common chord.

FIG. 5. For three given circles there are three radical axes. Two of them intersect at the point X. This point accordingly has equal powers with respect to the circles $O_1$ and $O_2$ as well as to $O_3$ and $O_2$; that is, with respect to all three. It is called the <u>Radical Center</u>. The line through X perpendicular to $O_1O_3$ is the radical axis of the two non-intersecting circles. This indicates a method of constructing the radical axis of two non-intersecting circles. Any circle such as $O_2$ will produce two lines intersecting on the radical axis of the two given circles.

FIG. 6. Construct the radical axes of the three circles, using only one auxiliary circle. Now draw the circle orthogonal to all three given circles. (Note: Two circles are orthogonal if their tangents at a point of intersection are perpendicular.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## THE NINE-POINT CIRCLE - EULER LINE - ORTHOCENTRIC SETS

FIG. 1. Locate the Orthocenter (intersection of altitudes); the Circumcenter (intersection of perpendicular bisectors of sides); and the Centroid (intersection of medians). These three points lie on a line called the Euler Line. (Johnson p.165)

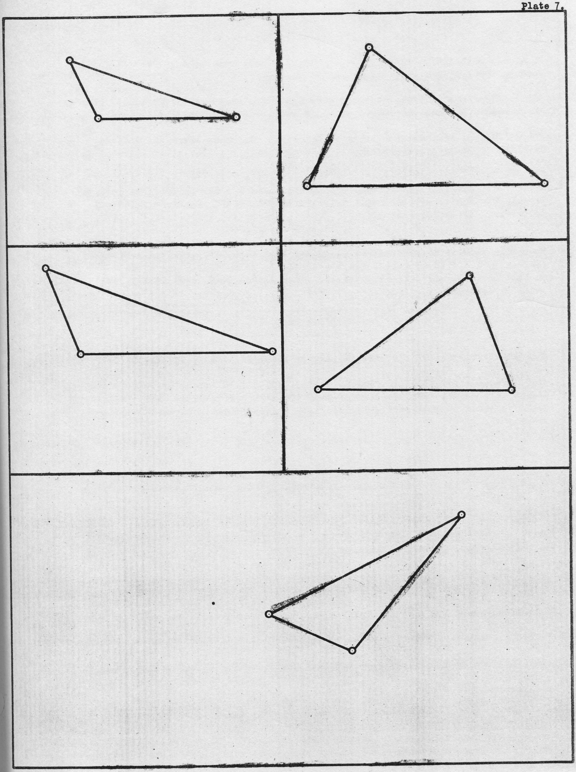
FIG. 2. Locate the midpoints of the sides; the midpoints of segments joining orthocenter to vertices; and the feet of altitudes. These nine points lie on a circle whose radius is half that of the circumcircle and whose center is midway between circumcenter and orthocenter. (Johnson p.195).

FIG. 3. Locate the orthocenter H. The four points, (the given vertices and the orthocenter H) form what is called an orthocentric set. Show that the four triangles formed from this (or any) orthocentric set all have the same Nine-Point circle.

FIG. 4. Draw the circumcircle and Nine-Point Circle. Verify that their internal and external centers of similitude are respectively the centroid and orthocenter of the given triangle. (Johnson p.197).

FIG. 5. Produce the sides of the Orthic Triangle (lines joining the feet of the altitudes) to meet the opposite sides of the given triangle. The three points thus formed lie on a line. (For proof, see Plate 26, 5).

Plate 7.



## REFLECTIONS

We assume that the path of a light ray or a billiard ball makes equal angles at a reflecting surface. This path generally is the shortest one possible.

FIG. 1. The shortest path from P to the line and then to Q is found by reflecting Q (or P) in the line and then joining the reflected point to P (or Q). Make the construction.

FIG. 2. Find the shortest path from P to one of the lines, then to the second, and then back to P. (Hint: Reflect P in each line).

FIG. 3. The Triangle of least perimeter that may be inscribed in a triangle ABC is the orthic triangle XYZ. This triangle makes equal angles with the sides of ABC. For, triangles BYA and CZA are both right triangles and thus

$$\angle HBZ = 90^\circ - A = \angle HCY.$$

Now a circle drawn on BH as a diameter passes through X and Z since BZH and BXH are right angles. Likewise, C, Y, H, X lie on a circle with CH as a diameter. In the first circle, $\angle$ ZBH and $\angle$ ZXH intercept the same arc and thus are equal. In the second circle, $\angle$ HXY and $\angle$ HCY are equal. Accordingly,

$$\boxed{\angle ZXH = \angle HXY}$$

Thus THE ALTITUDES OF A TRIANGLE BISECT THE ANGLES OF ITS ORTHIC TRIANGLE. (Schwarz, p. 345).

FIG. 4. Swimmers are to jump off a circular float, swim to shore #1, then to shore #2, and then back to their starting point. How would you pick the shortest path to win the race? (Hint: Draw a tangent to the circle that is perpendicular to the line joining center and intersection of shore lines).

FIG. 5. The billiard ball P is to touch the four successive cushions 1,2,3,4, and return to P. Draw the path. (Hint: Reflect P successively in the sides). Can you make a return shot on two adjacent cushions? On three?

FIG. 6. The billiard ball P is to touch all sides of the triangular table and return to its original position. What paths are possible?

FIG. 7. The Parabola is a curve such that any billiard ball such as P traveling parallel to the axis of symmetry will pass through a fixed point F, called the focus, and then be reflected along another parallel to the axis. List some properties of this curve and your reference.

FIG. 8. If the table is Elliptical, there are two foci, $F_1$ and $F_2$. If the ball P be shot along the line $PF_2$ it will pass through $F_1$ after reflection and then continue to travel alternately through the foci.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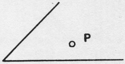


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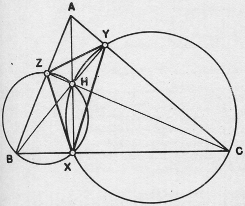


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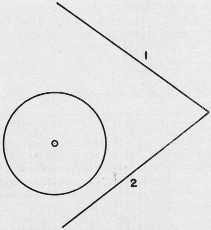


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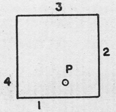


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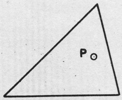


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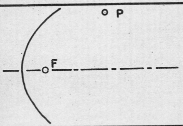


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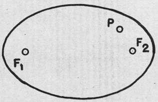


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## REGULAR POLYGONS

The discussion of regular polygons can be carried on conveniently if use is made of complex numbers. Such numbers are of the form $x + iy$ where $x$ and $y$ are real numbers and the letter $i$ represents the quantity: $\sqrt{(-1)}$. ($i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.) For pictorial purposes we shall agree to plot the point $(x,y)$ on a set of perpendicular axes as a representation of the number $z = x + iy$.

FIG. 1. In this discussion we need consider only those points which lie on the unit circle; that is, those for which $\sqrt{(x^2 + y^2)} = 1$. The inclination $\theta$ is found from $\tan\theta = y/x$. Thus:

$$x = \cos\theta, \quad y = \sin\theta \; ; \quad \text{and} \quad z = x + iy = \cos\theta + i\cdot\sin\theta.$$

FIG. 3. A surprising feature of these unit complex numbers is discovered on raising them to powers:

$$z = \cos\theta + i\cdot\sin\theta\,,$$

$$z^2 = (\cos\theta + i\cdot\sin\theta)^2 = \cos^2\theta + 2i\cdot\sin\theta\cos\theta + i^2\sin^2\theta = \cos 2\theta + i\cdot\sin 2\theta\,,$$

$$z^3 = (\cos\theta + i\cdot\sin\theta)^3 = (\cos 2\theta + i\cdot\sin 2\theta)(\cos\theta + i\cdot\sin\theta) = \cos 3\theta + i\cdot\sin 3\theta,$$

and generally:

$$\boxed{z^n = (\cos\theta + i\cdot\sin\theta)^n = \cos(n\theta) + i\cdot\sin(n\theta).}$$

The geometrical meaning of this should be apparent: If $z$ is such a number with inclination $\theta$, then $z^2$, $z^3$, $z^4$, etc., are all points upon the unit circle with inclinations $2\theta$, $3\theta$, $4\theta$, etc., respectively. It is just this property that makes them particularly useful to us since they represent points evenly distributed around the circle and thus are vertices of regular polygons. By setting $z^n = 1$ we demand $n$ such points with one of them at the unit point. It is then the Representative Equation of a regular polygon of $n$ sides.

This Representative Equation can always be factored into the form:

$$(z-1)\,(z^{n-1} + z^{n-2} + \ldots \quad z + 1) = 0.$$

The first factor equated to zero gives one vertex. The other $(n-1)$ vertices are given by the roots of the second factor. But it will not be necessary to solve this equation of degree $(n-1)$. For, since $z^n - 1 = 0$ can be rewritten (dividing by $z^n$) as $1/z^n - 1 = 0$, it is obvious that not only is $z$ a root but its recriprocal $1/z$ is also a root. Now, as may be verified by cross multiplication:

$$1/z^K = 1/(\cos K\theta + i.\sin K\theta) = \cos K\theta - i.\sin K\theta.$$

Therefore, since $z$ is any vertex and since $1/z$ is the reflection of $z$ in the line of real numbers, then all of our polygons will be symmetrical to this line and the resulting construction is considerably lightened.

FIG. 2. Since $z+1/z$ is a real number (the double of the abscissa of $z$ or the diagonal of the rhombus built on $0$, $z$, and $1/z$) we may employ the substitution:

$$\boxed{z + 1/z = 2x} \quad \text{from which:} \quad z^2 + 2 + 1/z^2 = 4x^2; \quad z^3 + 3z + 3/z + 1/z^3 = 8x^3; \text{ etc.,}$$

in order to aid in the algebraic solution of any Representative Equation. If a value $x$ can be determined and laid off, the corresponding vertex may be located by erecting the ordinate to meet the circle.

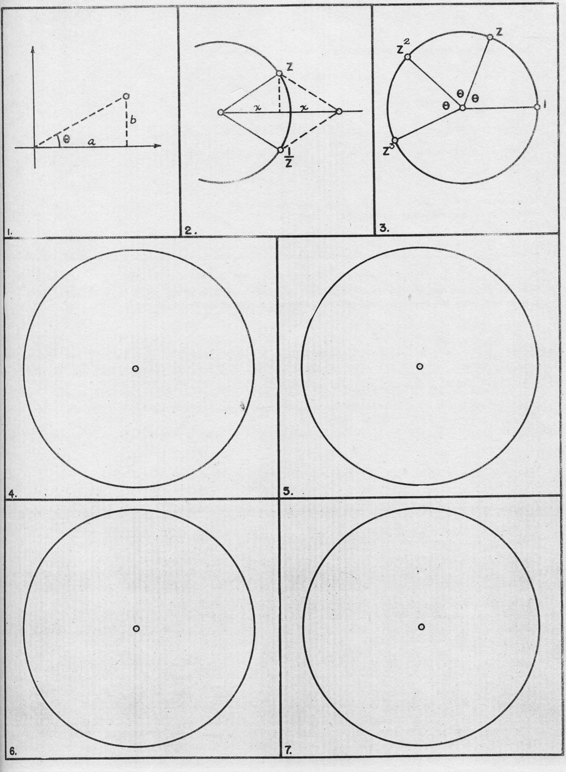
In the following, the student is required to solve each Representative Equation for $z$, using these values in the construction of the polygons, and calculate the length of a side, $S$, of each:

FIG. 4. The Triangle: $z^3 - 1 = 0$. $\qquad S_3 =$ ______________.

FIG. 5. The Square: $z^4 - 1 = 0$. $\qquad S_4 =$ ______________.

FIG. 6. The Hexagon: $z^6 - 1 = 0$ $\qquad S_6 =$ ______________. Notice that this equation includes the vertices of the Triangle.

FIG. 7. The Octagon: $z^8 - 1 = 0$ $\qquad S_8 =$ ______________. Notice that this equation includes the vertices of the Square.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# THE PENTAGON

The Pentagon has for Representative Equation:

$$z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1) = 0.$$

Writing the second factor as

$$z^2 + z + 1 + 1/z + 1/z^2 = 0,$$

we make the substitution: $z + 1/z = 2x$ (see Plate 9.3) and obtain

$$4x^2 + 2x - 1 = 0,$$

whose roots: $x = (-1 + \sqrt{5})/4$, $(-1 - \sqrt{5})/4$ are the abscissas of pairs of vertices of the Pentagon. From these values, calculate the length of a side:

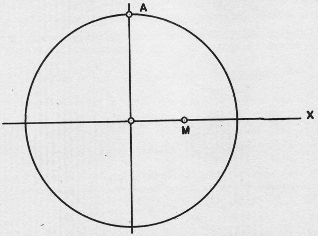
$$S_5 = \underline{\hspace{4cm}}$$

FIG. 1. Given the unit circle. Describe an arc with center at $M: (1/2, 0)$ and radius MA, cutting the X-axis in B. The length of its chord AB is equal to $S_5$, a side of the Pentagon. Why?

FIG. 2. Given the unit circle. The circle $4x^2 + 4y^2 + 2x - 1 = 0$ has its center at H: $(-1/4, 0)$ and passes through K: $(0, 1/2)$. The tangents to this circle at the points where it cuts the X-axis pass through the vertices of the Pentagon. Why?

FIG. 3. The point B: $(0, 1/2)$ is joined to $P_5$: $(1, 0)$. The bisector of $\angle\, OBP_5$ meets $OP_5$ in the abscissa of $P_1$, one vertex of the Pentagon. Establish this fact.

[Hint: $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$]

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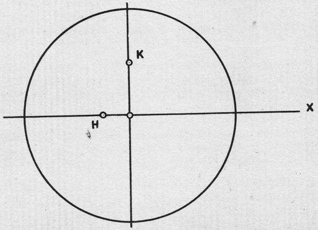


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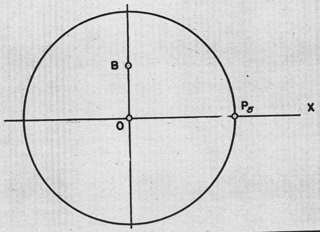


3.

## POLYGONS OF 7, 15, 17 SIDES

For the discussion to follow, we borrow a theorem from Algebra, stated without proof*:

IF AN EQUATION OF THIRD DEGREE WITH INTEGER COEFFICIENTS AND LEADING COEFFICIENT UNITY HAS NO INTEGER ROOT THEN IT HAS NO ROOT CONSTRUCTIBLE BY STRAIGHTEDGE AND COMPASSES.

<u>The Heptagon.</u> The Representative Equation for a regular 7-sided polygon is:

$$z^7 - 1 = (z - 1)(z^6 + z^5 + \ldots + 1) = 0.$$

The second factor: $z^3 + z^2 + z + 1 + 1/z + 1/z^2 + 1/z^3 = 0$, becomes, on substituting $z + 1/z = x$:

$$x^3 + x^2 - 2x - 1 = 0.$$

The roots of this equation are the double abscissas of pairs of vertices of the Heptagon. If this equation has an integer root, that root must be either +1 or -1 since on dividing by x:

$$x^2 + x - 2 = 1/x,$$

we see that no other integer could possibly satisfy the equality. Thus by the foregoing theorem there is no constructible root (since neither +1 nor -1 satisfies the equation) and the <u>Heptagon</u> is not constructible by straightedge and compasses.

FIG. 1. A simple straightedge and compasses <u>approximate</u> construction develops from the following. One-half the side of an equilateral triangle is $\sin 60^\circ = 0.86602$ (approx.). The side of the Heptagon is $2.\sin(180^\circ/7) = 0.86774$ (approx.). Thus, an error less than a thousandth part is committed in taking the side of the Heptagon as half that of the Triangle. Make the construction.

<u>The Enneagon</u> (9-sides) is represented by $z^9 - 1 = (z^3 - 1)(z^6 + z^3 + 1) = 0$, the latter factor of which reduces to:

$$x^3 - 3x + 1 = 0$$

on substituting $z + 1/z = x$. Here is another instance of an equation with non-constructible roots, and the Enneagon is therefore not constructible. Is it possible to trisect with straightedge and compasses an angle of $120^\circ$?

Regular polygons of 11 and 13 sides are also not constructible. Is the 14-gon? Give the lengths of a side of the Decagon (10) and of the Dodecagon (12):

$S_{10} =$ ________________. $S_{12} =$ ________________.

FIG. 2. <u>The Pentedecagon</u> (15-gon) is represented by

$$z^{15} - 1 = (z^3 - 1)(z^{12} + z^9 + z^6 + z^3 + 1) = (z^5 - 1)(z^{10} + z^5 + 1) = 0.$$

From an inspection of these factors, it is clear that its vertices include those of both the Triangle and Pentagon. The central angle subtended by each side is $24^\circ$. The Triangle and Pentagon are constructed with $\angle TOA = 120^\circ$, $\angle POA = 144^\circ$. Their difference, $\angle POT$ is $24^\circ$ and thus chord PT is the side of the regular Pentedecagon.

FIG. 3. Surprising indeed is the fact that the regular 17-gon can also be constructed by straightedge and compasses. The construction is given without proof.† Draw the perpendicular radii OA = OB = 1. Upon OB mark the point D:(0, -1/4). With two bisections locate the point E on OA such that $\angle ODE = \angle(ODA)/4$. Construct $\angle EDH = 45^\circ$. Draw the circle with AH as diameter meeting the line OB in K. With E as center and EK as radius draw the circle meeting AA' in L and M. Perpendiculars to AA' at L and M give the vertices $P_3$ and $P_5$ of the regular 17-gon. A side may then be found by bisecting $\angle P_5OP_3$ obtaining the point $P_4$.

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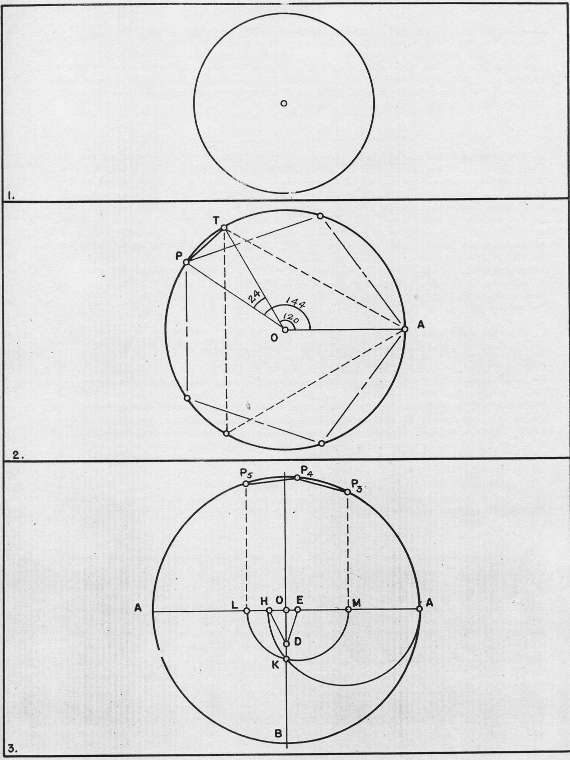
* See Dickson, p.33.

† See Richmond.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Polygons of $2^n$, $2^n.3$, $2^n.5$, $2^n.15$ sides. We have shown that the regular polygons of 3, 4, 5, 15 sides are constructible by straightedge and compasses. Since these tools are capable of bisecting any angle, it follows that regular polygons of 6, 8, 10, 12, 16, 20, etc., sides are constructible in the same sense. The fact that these polygons could be constructed was well known even to the early Greeks. However, other possibilities including the 17-gon were totally unsuspected for about two thousand years. Gauss in 1801 showed that there was a remarkable set of such constructible polygons: those the number of whose sides was a prime expressible in the form:

$$N = 2^{2^p} + 1.$$

Now many years before the time of Gauss, Fermat had considered numbers of this type and found that if p were given any of the values 0, 1, 2, 3, 4, then the resulting value N was indeed a prime. For some unexplained reason, he did not find out anything concerning the nature of N for p greater than 4. We know now that if p = 5, N is 4,294,967,297 **and this number is divisible by 641.** The labor involved in this calculation must have consumed hours and perhaps days. If the reader is somewhat skeptical, let him find the value of N when p = 6, that is, $N = 2^{64} + 1$, and then try to find a divisor of N. There is one. An idea of the magnitude of this number can be gotten from the story of the inventor of the chess game and his grateful king. As a reward, the king agreed to give the man one grain of wheat for the first square on the board, two grains for the second, four for the third, and so on, doubling the number each time. The total number of grains is exactly $2^{64} - 1$. Using a conservative estimate for the size of a single grain, a standard pint would contain 9,216 grains, a gallon 73,728, and the total would amount to 31,274,997,412,295 bushels. This is approximately 7,000 times the world production for the year 1935.

The extent to which the investigations of these Fermat numbers have been carried is amazing. But no one has been able to find a value of p greater than 4 which makes N a prime. It is definitely known that if p is any of the values:

5, 6, 7, 8, 9, 11, 12, 18, 23, 36, 73

the corresponding value of N is composite - that is, divisible by some number and thus not a prime. Nothing is known about the nature of N for p = 10, 13, 14, etc. That human beings and their machines (see Lehmer's factor machine at Lehigh) are capable of calculating and factoring such huge numbers borders on the miraculous. The number N for p = 36, for instance, is composed of more than 20 **trillion** digits. According to Lucas, the strip of paper upon which the number is written would encircle the earth. If we should print such an inconceivable number as N for p = 73 in volumes the size of the Encyclopedia Britannica, these books would overflow every library in every town and city of the United States.

A general constructibility rule, given without proof by Gauss, follows:
The only regular polygons that can be constructed by straightedge and compasses are those the number of whose sides can be expressed in the form:

$$N = 2^n.(2^{2^a} + 1)(2^{2^b} + 1)(2^{2^c} + 1) \ldots$$

where each number in parenthesis is itself a prime and any one of the lettered exponents may be zero, with $a \neq b \neq c$.

Number of Constructible Polygons. The reader has perhaps realized by this time that the totality of polygons which are constructible by straightedge and compasses is small compared to the totality of non-constructible ones. For N between 100 and 300, there are only 13 constructible polygons; from 300 to 1,000 sides, there are only 15 more; and from 1,000 to 1,000,000 sides, only 154 altogether. The chance of naming at random a constructible polygon of less than a million sides is thus about one in five thousand.

REGULAR POLYGONS

Those constructible polygons, the number of whose sides is less than 100, are listed in the following table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 | 4 | 5 | 6 | | 8 | | 10 |
| | 12 | | | 15 | 16 | 17 | | | 20 |
| | | | 24 | | | | | | 30 |
| | 32 | | 34 | | | | | | 40 |
| | | | | | | | 48 | | |
| 51 | | | | | | | | | 60 |
| | | | 64 | | | | 68 | | |
| | | | | | | | | | 80 |
| | | | | 85 | | | | | |
| | | | | | 96 | | | | |

L. E. Dickson has given formulas by means of which the total number of constructible polygons below

$(2^x + 1)$ sides can be determined:

If $x$ is less than 32, the number of constructible polygons is

$$(x - 1)(x + 2)/2;$$

If $x$ is greater than 32 but less than 128, the number of such polygons is

$$(32x - 497).$$

Make a list of the constructible polygons with number of sides between 100 and 300 in a table below:

## SECTION II

## DISSECTION OF PLANE POLYGONS

Among the demands of Euclid we find that Polygons, particularly triangles, must be proved congruent by superposition. Only after this is done initially do we notice that congruence may be established by inference. For the purposes of this section we shall assume the ability to transform into a cut any straight line that has been previously constructed by straightedge and compasses.

The maximum value of the methods indicated here can be obtained only by making models, preferably with colored cardboard, to illustrate. Even to someone little acquainted with geometry, the jig-saw game of fitting together the pieces to form the polygons of equal area will prove diverting and stimulating. Place your models in envelopes for future reference. Models made of masonite with the pieces joined by small strap hinges make excellent illustrations. For cardboard models, use a photo trimmer.

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|---|---|
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# DISSECTION OF PLANE POLYGONS

FIG. 1. Given the scalene triangle ABC. A cut is made through the midpoints X, Z, of two sides. The two pieces will fit together to form a parallelogram in two ways, having, of course, the same area as ABC. Why is XZ parallel and equal to half AC?

FIG. 2. The scalene triangle ABC is cut along the lines joining the midpoints of its sides into four pieces. These pieces are all congruent. Why?

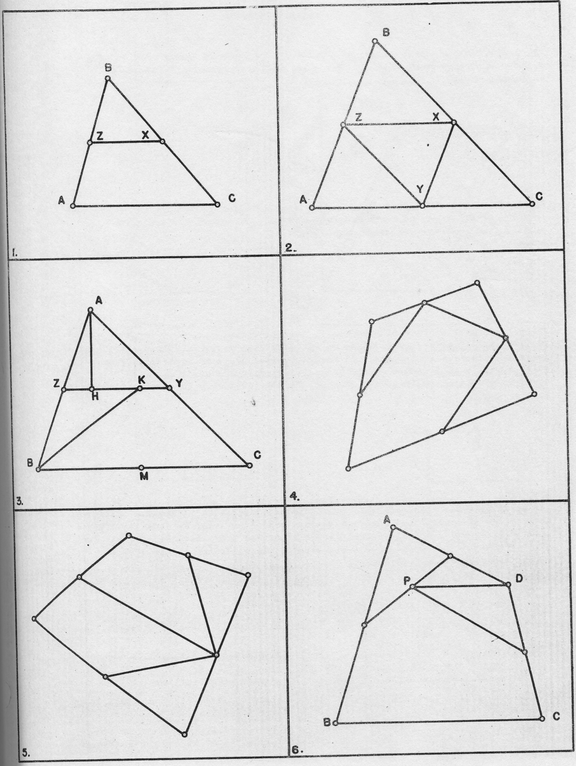
FIG. 3. The scalene triangle ABC is cut into four pieces along the line ZY, where Y and Z are the midpoints of AC and AB; along AH, the perpendicular to YZ; and along BK, where MK is the perpendicular bisector of BC. Fit these pieces together to form successively a parallelogram, a rectangle, and a right triangle. Explain.

FIG. 4. The lines joining in order the midpoints of the sides of any quadrilateral form a parallelogram. Why? Find the area of this parallelogram and thus show that the area of any quadrilateral is half the product of its diagonals and the sine of their included angle.

If cuts are made along three of these lines as shown, the four pieces may be fitted together to form a parallelogram with area equal to that of the quadrilateral. Explain.

FIG. 5. From the preceding discussion we find a method of dissecting any quadrilateral into four pieces which will form a triangle: From the midpoint of any side, cut to the midpoints of the other three. Rearrange these to form the triangle and explain.

FIG. 6. Here is shown another method of reducing the quadrilateral to a triangle. Cut along the line joining the midpoints of AB and AD. Now cut through D parallel to BC obtaining the point P. The third cut is from P to the midpoint of CD. Explain and discuss the possible failure of this method.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THE THEOREM OF PYTHAGORAS

FIG. 1. The square of side $(a + b)$ may be dissected into the four right triangles with legs $\underline{a}$ and $\underline{b}$ as shown and the inner square of side $\underline{c}$. Thus, since the four triangles form two rectangles of dimensions $\underline{a}$ and $\underline{b}$:

$$(a + b)^2 = 2ab + c^2 \qquad \text{or} \qquad a^2 + b^2 = c^2,$$

the <u>Theorem of Pythagoras</u>.

FIG. 2. The Theorem of Pythagoras may be demonstrated by the following dissection. The square ABCD is of side $\underline{c}$. Cut it into the four right triangles APD, AIB, CND, and CMB with sides $\underline{a}$ and $\underline{b}$. The smaller central square then has its side equal to $(a - b)$.
Thus

$$c^2 = 2ab + (a - b)^2 = a^2 + b^2.$$

FIG. 3. From the point of view of dissection, the Theorem of Pythagoras is a clue to the process of <u>adding two squares to make one</u>. To dissect and add, place the given squares, ABCD and AEFG, so that two of their sides form the legs of a right triangle as shown. Make a cut in the larger square from B perpendicular to the hypotenuse followed by the cut perpendicular to that as shown. In the smaller square cut from G perpendicular to the hypotenuse. These five pieces will form a single square. Explain.

FIG. 4. An alternate and extremely simple addition is the following elegant dissection. Place the two given squares so that a right triangle is formed as shown. Cut through the center of the larger square along lines parallel and perpendicular to the hypotenuse. This produces four <u>congruent</u> pieces which may be reassembled at the corners of the sum-square leaving a center hole into which the smaller square may be fitted. Explain.

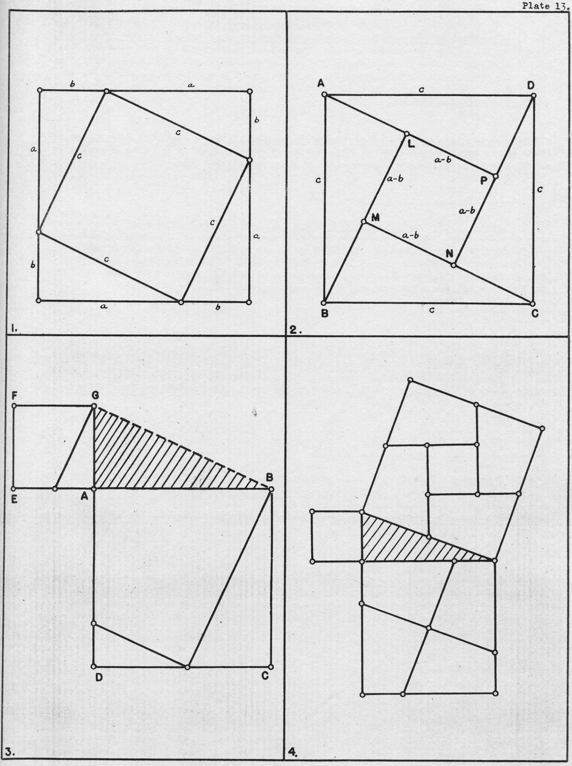
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FIG. 1. To transform the given parallelogram ABCD, with sides $a$, $b$, and angle $\theta$, to another parallelogram of specified shape. Suppose the two sides of the required parallelogram are $x$ and $y$. Locate the point X in BC such that $AX = x$; then construct $DY = y$. Cuts are made along AX and DY to give three pieces which will form the required parallelogram. For, no matter where X is selected on BC, triangle AXD has fixed base and constant altitude equal to that of the given parallelogram. Thus its area is half that of the parallelogram and accordingly,

$$ab.\sin\theta = x.y.\sin\phi.$$

FIG. 2. Here is given a second method of reducing a given parallelogram to a specified parallelogram. Let BX be constructed equal to a desired side. The line AX produced to meet DC in W gives DW as the second side. Cut along AX and YZ where triangle AYZ equals triangle XWC. Explain.

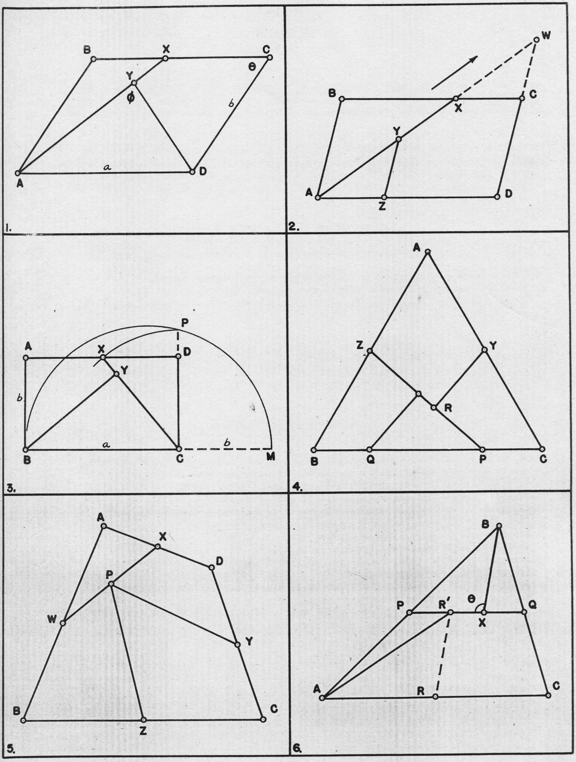
FIG. 3. To reduce a given rectangle ABCD with sides a and b to the square to equal area. Since the area of the required square is ab, its side is $\sqrt{(ab)}$. We must thus construct a cut whose length is the mean proportional between a and b. Accordingly, locate the point M so that MB equals $(a + b)$. On this as a diameter construct a semicircle. Extend CD to P. Then $PC = \sqrt{(ab)}$. Swing this off with C as center locating the point X. Cut from X to C, then from B perpendicular to CX. Show that $BY = \sqrt{(ab)}$. (In case Y does not fall within the rectangle, more cuts are required.)

FIG. 4. A famous dissection problem is that of transforming an equilateral triangle into the square of equal area. Let the triangle have side equal to 2. Then its area (and that of the required square) is $\sqrt{3}$. Let Z, Y be the midpoints of AB and AC respectively. With a constructed length $\sqrt[4]{3}$ as radius, describe an arc with center at Z which cuts BC in P. Locate Q so that $PQ = 1$. Cut along ZP, then from Y and Q perpendicular to ZP. These four pieces form the square. Why?

FIG. 5. A reversion of the dissection of Fig. 4 is the more general reduction of a quadrilateral ABCD to a specified triangle. Let X, Y, Z, W be the midpoints of the sides. The parallelogram XYZW is half the area of the given quadrilateral. Thus, for any point P on XW, triangle PYZ is one-fourth the area of the quadrilateral. If P be selected such that PY and PZ are equal, cuts along XW, PY, and PZ will reduce the quadrilateral to an isosceles triangle, whose base is the length of the diagonal BD. Explain.

FIG. 6. To transform a given triangle ABC to another triangle with the same base, AC, and a specified angle $\theta$. Cut along PQ where P and Q are the midpoints of two sides, then along BX where angle $BXP = \theta$, then along AR' where RR' is parallel to BX and R is the midpoint of AC. These four pieces form the triangle with the same base and specified angle $\theta$. Explain and discuss the possibility of the dissection yielding more pieces.

How does the dissection of Fig. 6 apply to the reduction of a polygon of n sides to one of $(n - 1)$ sides?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# SECTION III

## THE COMPASSES

(Geometry of Mascheroni)

It is proved here that the entire plane geometry of Euclid may be effected by means of the compasses alone. This was the important contribution of Mascheroni, a protege of Napoleon, in 1797. Recent disclosures, however, indicate that Mascheroni was anticipated about one hundred years by Georg Mohr.

It should be noted that although the straight line as a whole cannot be constructed by compasses alone, yet an infinitude of arbitrary points may be located upon it. It is remarkable indeed that the points of intersection of two such hypo-lines, given only by two pairs of points, may be determined solely by the compasses. The fact that it is capable of producing all plane euclidean constructions is established by finding the intersection of:

1. two circles (which is immediate)
2. a hypo-line and circle (Plate 18)
3. two hypo-lines (Plate 18).

The idea of inversion is introduced here primarily as a service. However, the subject is of interest in itself and reappears in Section VI with a mechanical interpretation.

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## THE COMPASSES

FIG. 1. With a single opening of the compasses draw a set of circles whose intersections form a rectangular network of points.

FIG. 2. Construct the reflection of P in the hypo-line* AB. (Hint: Use A, B as centers).

FIG. 3. Construct other arbitrarily selected points on the hypo-line AB. (Hint: Select any point P not on the line and reflect. Then use arbitrary radii).

FIG. 4. Find the point C collinear with A and B such that AB = BC. (Hint: Recall the Hexagon).

FIG. 5. Draw members of the family of circles all of which pass through A and B.

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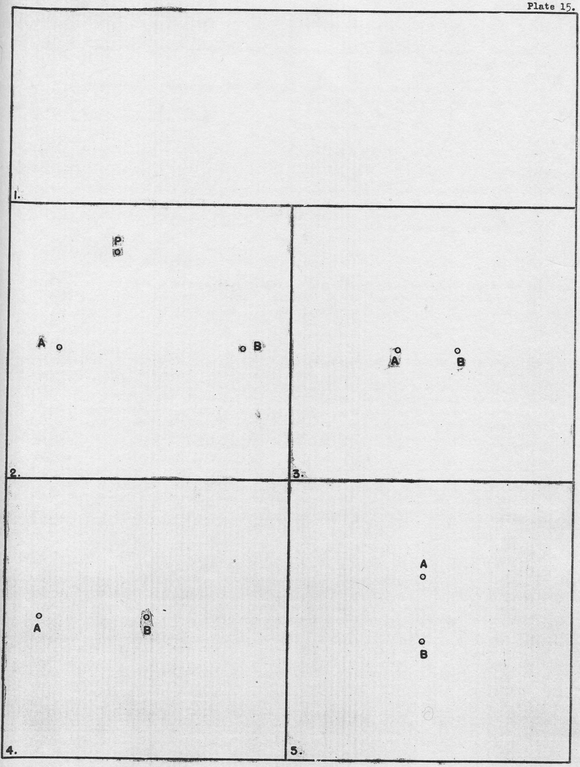
* "Hypo" is a contraction of the word "Hypothetical".

Plate 15.



# INVERSION

The Compasses is a natural instrument for the interpretation of a fundamental study of plane geometry called Inversion.

FIG. 1. Given two selected points, O, A, at a distance $a$ apart. Two other points, U, V, are inverse to each other with respect to OA if:

1. they are collinear with O and A,

and 2. $(OU)(OV) = (OA)^2 = a^2$.

FIG. 2. This idea of inverse points may be enlarged to include inverse curves: If V travels along a specified curve, its inverse U travels along a corresponding curve, and the point A traces the circle with radius a. We shall call this circle the base circle. Locate the points that are inverse to themselves.

If we take the point O as the origin of a system of polar coordinates and an arbitrary line as polar axis so that $OU = r$, $OV = s$, then

$$\boxed{r.s = a^2}$$

expresses the condition for inverse points.

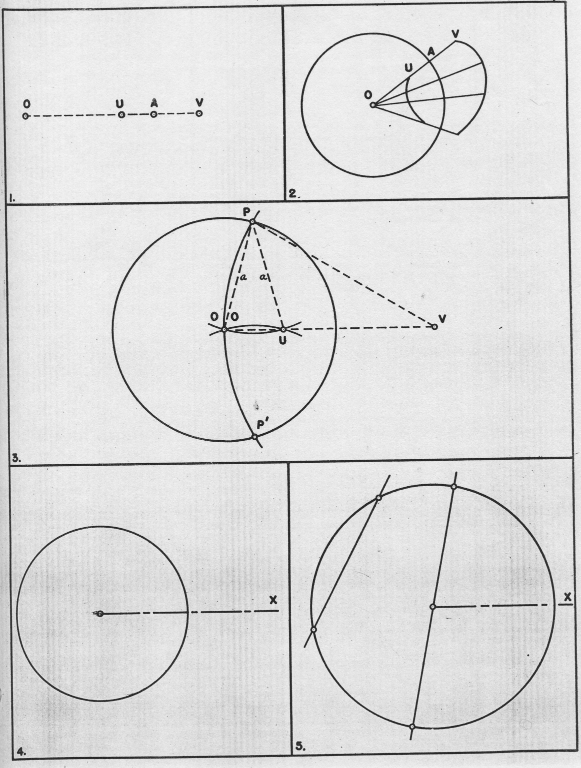
FIG. 3. To find the inverse of a given point V with respect to the circle O(a), draw an arc with center at V passing through O and meeting the base circle in P and P'. With P and P' as centers, draw arcs of radius $a$. These meet at O and again at U, the inverse of V. For the proof, consider the lines from P to O, U, and V. Triangles OPU and PVO are isosceles and similar since they have a common base angle POV = $\phi$. Thus the proportion:

$$OU/a = a/OV, \qquad \text{or} \qquad (OU)(OV) = a^2.$$

FIG. 4. The base circle is given with radius 1. Using its center as the pole, draw the circle (which passes through the pole): $4r = 3.\cos\theta$. Invert several points of this circle and give the polar equation of the inverse figure.

What is the inverse of the circle: $r^2 = A.r.\cos\theta + B.r.\sin\theta + C$ ?

FIG. 5. Given the lines $\tan\theta = K$ and $A.r.\cos\theta + B\ r.\sin\theta = C$. Construct several points on their inverse curves and give their inverse equations.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# INVERSION

FIG. 1. Given the points A, B. Locate their midpoint. (Hint: Find C such that AB = BC, then invert in A(B)*) How many circles need be drawn to solve the problem?_______________ How many radii?_______________.

FIG. 2. Draw the circle $r^2 - (5/2)r.\cos\theta + 1 = 0$ which is orthogonal to the base circle, $r = 1$. What is its inverse with respect to this base circle?________________________. What statement can you make regarding the inverse of any circle orthongal to the base circle?

FIG. 3. Plot points upon the Lemniscate $r^2 = \cos 2\theta$. Notice that the curve is tangent to the unit base circle. Locate points upon the inverse of this curve and give its polar and rectangular equations:

The lines tangent to the Lemniscate at the origin are perpendicular. Into what do these invert and what is their relation to the inverse curve?

A polar equation of the Limacon is $r = a + b.\cos\theta$. What are the rectangular and polar equations of its inverse?

In the following use $r = 1$ as the base circle:

FIG. 4. Plot points upon the Limacon and its inverse for $a = 1$, $b = 1/2$. Identify.

FIG. 5. Plot points upon the Limacon and its inverse for $a = 1$, $b = 1$. Identify.

FIG. 6. Plot points upon the Limacon and its inverse for $a = 1$, $b = 2$. Identify.

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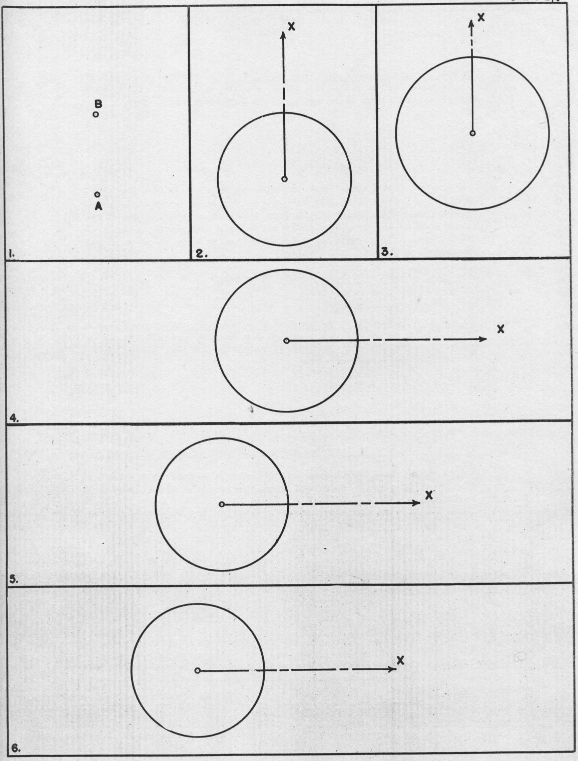
* A(B) is the circle with center A passing through B.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THE COMPASSES

FIG. 1. To draw the circle through three given points, A, B, C. With center at A draw the circle through B. Invert C in this circle, obtaining the point C'. The hypo-line BC' is the common chord of the base circle and the desired circle. Why?

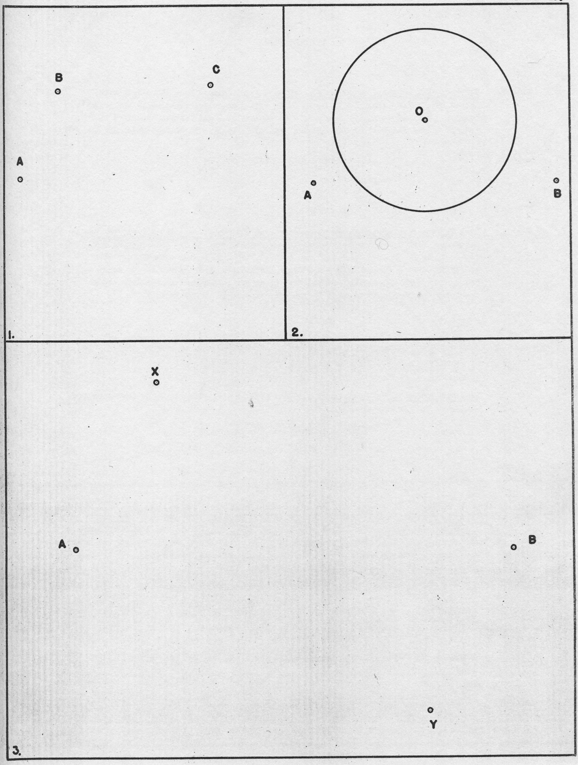
Now reflect A in this common chord, BC', obtaining point A'. We have then, AB = A'B. Invert A' in the same circle A(B), obtaining O. This is the center of the desired circle. For, B', the reflection of B in AO, is the remaining intersection of the common chord and the circle of inversion and is thus a point on the desired circle. But the point O, by construction, is equidistant from B, A, and B' and therefore is the desired center.

FIG. 2. To find the intersections of the hypo-line AB and the given circle, simply reflect the center O in AB and with that point as center, describe the circle with the same radius as the given circle. These two circles have AB as radical axis and their intersections are the desired points.

FIG. 3. To determine the intersection of two hypo-lines given by the pairs of points A,B; and XY. Using an arbitrary circle of inversion, both lines may be inverted into circles through the center of inversion. These circles intersect in one further point, P, whose inverse is the desired intersection of the hypo-lines. Make the construction.

REMARKS: As discussed in preceding plates, the constructions possible by straightedge and compasses are but combinations of the three fundamental ones: the intersections of two circles; the intersections of a circle and a line; and the intersection of two lines. We have shown by the constructions of this plate that the plane geometry of Euclid may be executed by means of the compasses alone.

Plate 18.



THE COMPASSES

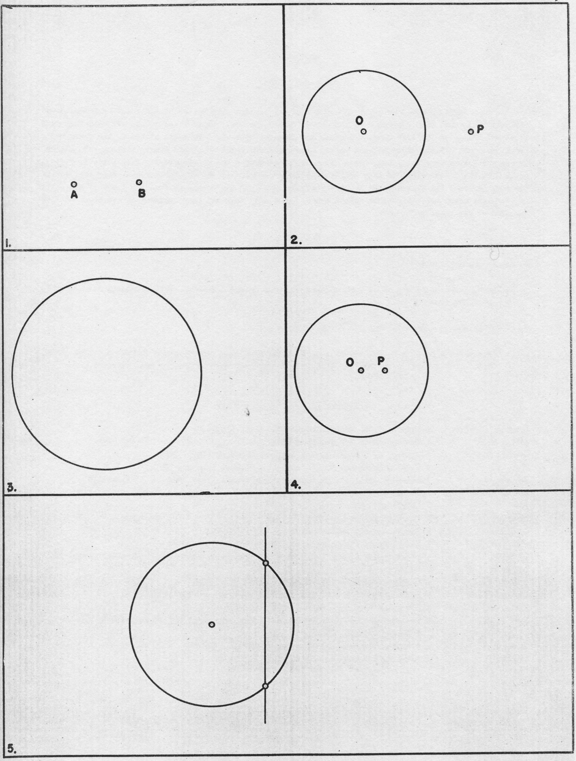
FIG: 1. Construct the vertices of a square on AB as a side. (Hint: Draw the circle with radius AB, center at B. Let A, G, H, C, be four consecutive vertices of the inscribed hexagon. With AH as radius draw arcs with centers at A and C which intersect in P. Then BP is the diagonal of the required square). Complete the construction and explain.

FIG. 2. Find the intersections with the given circle of the hypo-line joining P and the center O. (Hint: With P as center and an arbitrary radius, draw an arc intersecting the given circle in A, B. Draw circles A(O) and B (O). With O as center draw the arc with radius AB cutting circles A(O) and B(O) in D and E. With DB as radius and centers at D and E, draw arcs intersecting at G. With OG as radius and D as center draw an arc intersecting the given circle at F, the point desired). Prove this.

FIG. 3. Find the center of the given circle. (Hint: Draw an arbitrary circle of inversion with center on the given circle).

FIG. 4. Find the inverse of the point P where OP is less than half the radius of the circle of inversion. (Hint: Let the radius of inversion equal 1. There is no loss of generality in assuming OP > 1/4. Find Q such that OQ = 2(OP). Then invert Q obtaining the point R. Locate S such that OS = 2(OR). S is the required point. Explain.

FIG. 5. Construct the circle of inversion for which the given line and circle are inverse figures.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## SECTION IV

## FOLDS AND CREASES

In creasing a sheet of paper, a point A of one portion (the upper) of the sheet is folded over and held coincident with a point B of the other (under) portion. While these points are held fast with thumb and finger of the left hand, the thumb and finger of the right hand are placed on the upper and lower portions. If the hands are now pulled apart with the right thumb and finger sliding, the points (upper and lower) upon which they slide are equidistant from A and B. Eventually this leads to a single point C on the crease which is thus equidistant from A and B. As the thumb and finger form this crease the tension keeps the distances on the two portions equal and the crease is thus the locus of all points of the sheet which are equidistant from A and B. Obviously this can be considered as the straight line bisector of the segment AB.

### POSTULATES

We assume the ability to:

I. Place one point of the sheet upon another and thus create a crease. This crease is assumed to be a straight line.

II. Establish the crease through two given distinct points.

III. Place a given point upon a given line so that the resulting crease passes through a second given point.* (This implies the ability to fold a crease over upon another or upon itself).

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* We assume the point and line are so situated that this may be accomplished. See Plate 22, 5.

## FOLDS AND CREASES

In the following the student is required to make the folding constructions, using small* pieces of paper with irregular edges and paste them (so they may be unfolded) on the opposite page. Thin colored art paper (an inexpensive package) provides excellent illustrations.

-----------o-----------

1. Given a point P and a line L. Establish the crease through P perpendicular to L.

2. Given a point P and a line L. Establish the crease through P parallel to L.

3. Form two perpendicular creases. Then with scissors cut the edge into a plane curve. When unfolded, the curve is symmetrical to both creases.

4. Select (crease) a triangle ABC with right angle at B and obtain the foot, D, of the altitude from B. Show that D and C are inverse points with respect to the circle whose center is A and whose radius is AB.

5. Select the scalene triangle ABC and obtain the foot D of the altitude from B. Fold the vertices A, B, C, over to D. From this model, show:

   (a) The sum of the angles of a triangle is $180^{\circ}$.

   (b) The line joining midpoints of two sides is parallel and equal to half the third side.

   (c) The area of a triangle is half the product of a side and its altitude.

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* About 8 or 9 square inches.

## FOLDS AND CREASES

Make models of the following and paste on the opposite page. In each construction make note of the postulates used.

-----------o-----------

1. Select (crease) a triangle and obtain its incenter (intersection of angle-bisectors).

2. Select a triangle and obtain its orthocenter (intersection of altitudes).

3. Select a triangle and obtain its circumcenter (intersection of perpendicular bisectors of sides.)

4. Select a triangle and obtain its centroid (intersection of medians).

5. The Parabola is the locus of points equidistant from a given fixed point called the Focus and a given fixed line called the Directrix. A familiar property is that any tangent bisects the angle between the focal radius and the line from the point of tangency perpendicular to the directrix.
   Thus, given a line L (a crease or edge) and a point F (a corner for instance) in a sheet of paper, move F along L and form the creases. Those creases are all tangent to the Parabola having F as focus and L as directrix. They are said to "envelope" the curve. Explain.

   (Note that Postulate III grants the ability to establish the crease through a given point that is tangent to the parabola defined by the line and the other point. If the first point lies within the parabola the construction is impossible).

## FOLDS AND CREASES

Make models of the following and paste on the opposite page. In each construction make note of the postulates used.

1. Establish a Square by creasing. Then fold the corners to the center and crease. These creases form a square inscribed to the first. Continue this to illustrate the sequence:

$$1,\ 1/2,\ 1/4,\ 1/8,\ 1/16,\ 1/2^n.$$

2. From a selected square, fold and crease the inscribed square. Find the intersections of creases bisecting the angles between the sides of the inner and outer square. These intersections are vertices of a Regular Octagon. Explain.

3. Crease the quadrisectors of the angles of a square and thus obtain the vertices of a Regular Octagon. Explain and compare areas.

4. Establish an Equilateral Triangle by creasing. (Hint: Obtain the perpendicular bisector of a selected segment).

5. Fold the corners of a equilateral triangle to the center and obtain the Regular Hexagon. Compare areas of the two polygons.

6. Refer to Plate 10 and crease the Regular Pentagon by some method given there.

## FOLDS AND CREASES

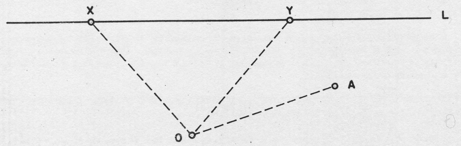
FIG. 1. Intersections of a given line and hypocircle. The intersections of the crease L and the hypocircle O(A) are found at once through Postulate III by folding OA so that A lies upon L at the points X and Y.

FIG. 2. The intersections of two hypocircles O(K) and O'(A') are found by first establishing their radical axis. Proceed as follows. Transfer the radius OK to OA along a parallel to O'A'. Since lines joining extremities of parallel diameters, AOB and A'O'B', meet in a center of similitude, (see Plate 5,1) the creases AA' and BB' meet in this point E. The crease EA meets the second circle again in C' (found by folding the crease perpendicular to BE through B'). Crease EB meets the first circle in D.

Now A, D, B', C' form a quadrilateral with right angles at a pair of opposite vertices. These four points thus lie on a circle, Z, which cuts the circle O(A) in A, D; and the circle O'(A') in B', C'. Thus creases AD and B'C' are the common chords of Z and O(A); and of Z and O'(A'). Accordingly, (see Plate 6,5) AD and B'C' meet in X, a point of the radical axis of O(A) and O'(A'). The crease through X perpendicular to OO' is this radical axis.

Having thus established the radical axis, its intersections with either circle, Fig. 1, are the required points.

REMARKS: WE HAVE PROVED THAT, UNDER THE CHOSEN POSTULATES, ALL CONSTRUCTIONS OF PLANE EUCLIDEAN GEOMETRY CAN BE EXECUTED BY MEANS OF CREASES.

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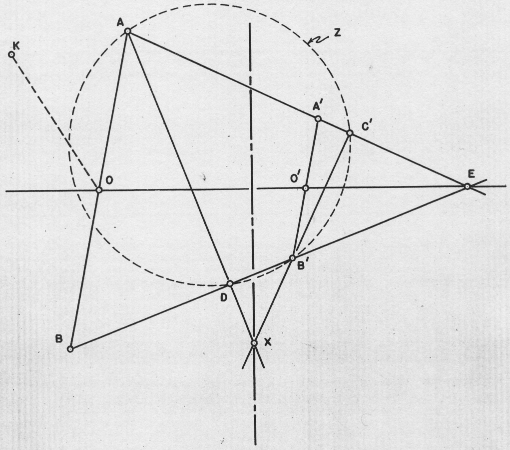



2.-

## KNOTS

The possibilities of folding and creasing may be extended if, in addition to the foregoing, we admit a process of knotting a paper strip whose edges are parallel. (The mathematical definition of "tightness" is difficult to state and it is hoped that the reader will not be confused over the meaning of the word).

In each of the following, knot the regular polygons and paste your models in the spaces provided. Half-inch strips cut the length of a standard sheet are serviceable.

Neither the Triangle nor Square can be formed into a knot that is strictly tight or self-supporting.

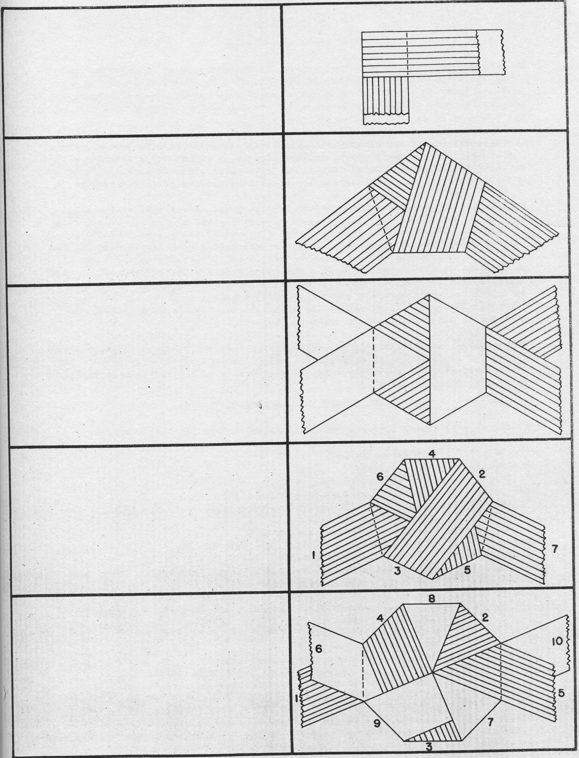
FIG. 1. A Square may be formed with two strips of the same width. Fold each over upon itself and insert an end of one strip into the fold of the other.

FIG. 2. To form the Pentagon, tie an over-hand knot in a single strip. (The over-hand knot is the first knot in tying a shoe string.) After tightening and creasing flat, unfold and consider the set of trapezoids formed by the creases.

FIG. 3. To form the Hexagon, tie the sailor's reef or square knot with two strips of the same width as shown. (Tuck the ends of each strip into the loop of the other.)

FIG. 4. To form the Heptagon, (not constructible by straightedge and compasses) use a single strip to knot the Pentagon. Before tightening, however, pass the lead under the knot and then through it as shown. Or, carry the lead through the sequence of numbers indicated. After tightening, unfold and examine the trapezoids formed by the creases. Locate these trapezoids in the given figure.

FIG. 5. To form the Octagon, first tie a loose over-hand knot in one strip, here the striped one going from 1-2-3-4-5. With a second strip of the same width, start at 6 passing over 1-2 and 3-4. Bend (do not crease) up at 7, passing under 4-5 and 1-2, and bending up again at 8. Pass under 3-4, over 1-2, and under 6-7. Bend up at 9 and pass over 3-4, under 7-8 and 4-5, emerging at 10.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# SECTION V

## THE STRAIGHTEDGE

(Synthetic Projective Geometry)

The instrument considered here has but one straight edge upon which there are no graduations. Following Euclid, its use is restricted to drawing a line of indefinite length through two given or established points. We have thus no medium of measurement and the familiar notions of distance, angle, area, parallelism and the like cannot be interpreted. Our only ability will lie in the identification of a line on two points and the point on two lines.

Although quite useful as an auxiliary instrument in general construction work, the Straightedge certainly does not appear very powerful. Surprising, however, is the fact that it is capable of solving complicated and elaborate problems of construction. An example of this is the remarkable construction of the tangent to any given conic (including the circle) from an external point. (See Plate 3, 13).

The Projective Geometry of Desargues, de la Hire, Poncelet, Steiner, Pascal, and Plücker developed in part through a need for a more descriptive element in painting and etching. More particularly, this and other non-Euclidean geometries arose as a result of repeated failures to prove the Euclidean postulate of parallels. The growth of the subject was experienced in two phases: the first beginning properly with the noteworthy treatment offered by Desargues and the far reaching theorem of Pascal; the second was the revival occasioned by the publication by Poncelet in 1822 of his famous notes made during a military imprisonment in Russia. A considerable time about the middle of the 19th century was spent in an academic war between those who advocated synthetic treatment entirely (the pure geometers) and those who believed that the only approach was through the medium of analysis. The result was far from injurious either to analysis or to geometry.

The principle of duality was received with the heartiest of welcomes. With the establishment of this principle "an already vast empire of geometry was doubled in extent".* Sylvester remarked that with the Plücker coordinates, geometry need no longer stumble along on one foot, that it could stride forth firmly on its two equal supports.

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* E. T. Bell in The Development of Mathematics, New York (1940) 316.

## THE STRAIGHTEDGE

A field in which the Straightedge plays a natural role is that of Projective Geometry, a product of Dürer, da Vinci, and others following the 15th century. The definition of some terms follows:

FIG. 1. A Range of Points is a set of points lying upon a line. The line is called a Carrier.

FIG. 2. A Pencil of Lines through a point consists of all lines that may be drawn through that point. We speak of them as "lines on a point".

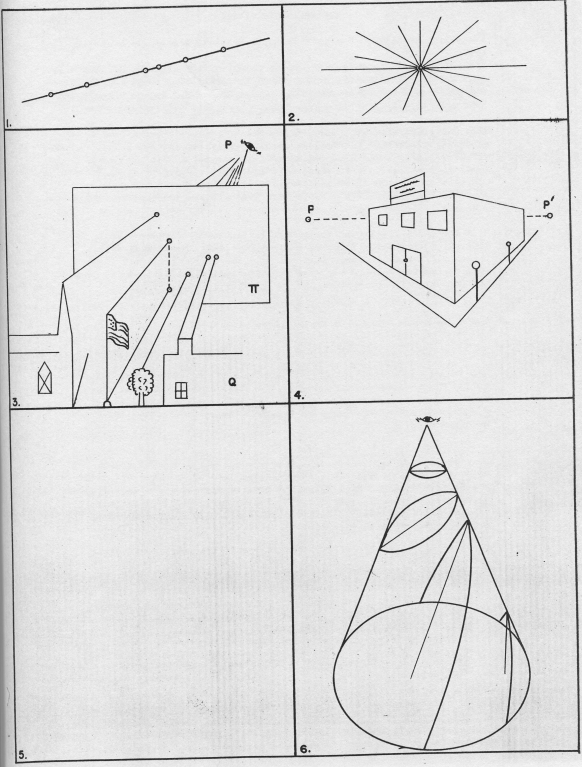
FIG. 3. Defines Projection. Lines drawn from a point P (the eye, for instance) to the various points of a set, Q, are cut by a plane $\pi$. The points of intersection of $\pi$ and these lines are called the projections of the points of Q onto the plane. It should be clear that lines project into lines and the intersection of two lines projects into the intersection of the two projected lines. Why?

Why is it advantageous to have two eyes? (Recall the old fashioned stereoptican).

FIG. 4. This geometry, as indicated in Fig. 3, seems then to be the geometry that occurs in photography. The parallel railroad tracks may appear to meet or to vanish at a certain point; rectangles may appear as parallelograms; circles may appear as ellipses; lengths are destroyed by projection. The quality that does not change is called an invariant and it is this phase of the matter that is of interest. The parallel edges of the building in Fig. 4. intersect at P and P', called vanishing points. The line PP' is, naturally, the horizon.

FIG. 5. From a magazine, select a photograph illustrating the principles of projection. Paste it in the space provided here. Some early artists were ignorant of this representation and as a result created flat and shallow paintings. Illustrations will not be hard to find.

FIG. 6. The plane sections of a circular cone are the Circle, Ellipse, Parabola, and Hyperbola. By placing the eye at the vertex of the cone these curves all appear alike. Thus in Projective Geometry there is no distinction among them and they are all indicated by the single term: conic. Most of their characteristics studied in Analytic Geometry are measurable properties. These will not appear as such in the present section. However, certain of their features of a projective nature will come to light in our investigations and these may later be translated into metric terminology.


1.
2.
P
π
Q
3.
P
P′
4.
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6.

FIG. 1. A cornerstone of Projective Geometry is Desargues' (dā-zargs) Theorem which is credited by Pappus to Euclid:

IF TWO TRIANGLES, 1, 2, 3, and 1', 2', 3', ARE IN SUCH POSITION THAT THREE LINES JOINING THEIR SEVERAL VERTICES MEET IN A POINT P THEN CORRESPONDING SIDES* PRODUCED MEET IN THREE POINTS X, Y, Z, WHICH ARE COLLINEAR. The line XYZ is called the axis of perspective, P the center of perspective, and the two triangles are said to be in perspective from P.

PROOF: Consider the situation in space. We have then the pyramid with vertex P cut by the two planes 1,2,3 and 1',2',3'. These two planes (generally) meet in a line L. Lines 1,2 and 1',2' lie in a plane through P and thus intersect in some point Z. This is a point of L since these same lines lie in the planes 1,2,3 and 1',2',3'. Similarly for X and Y. The whole space configuration can now be projected to a plane; points going into points and lines into lines with no change in our results. Thus the theorem is established and L is the axis.

FIG. 2. As an application, draw the line through X and the inaccessible intersection of two given lines, a, b. (Hint: Through X draw two arbitrary lines meeting a and b in 3 and 3', respectively. Select an arbitrary point P on 33' as the center of perspective. A second arbitrary line from P meets X,3 in 2 and X,3' in 2'; a third line from P meets a in 1 and b in 1').

FIG. 3. Draw a line through X parallel to the two given parallel lines.

FIG. 4. Given two pairs of lines a,b and x,y which have a pair of inaccessible intersections. Draw the line through these intersections. (Hint: Select a point of perspective upon the diagonal of the given quadrilateral, using a,x and b,y as corresponding sides of perspective triangles. This locates one point of the desired line. A repetition completes the construction).

FIG. 5. Lines drawn from the vertices of a triangle to a point P meet the opposite sides in A', B', and C'. The triangle of these latter points is called the pedal of P. Show that the sides of any pedal triangle meet the opposite sides of the original triangle in three collinear points. Make special mental note of this for P as the circumcenter, the incenter, the orthocenter, etc. (See Plate 4). Discuss carefully the situation as P is allowed to approach the position of the centroid.

Additional Problems: Verify that three given lines with an inaccessible intersection are concurrent.

A given line ends at an obstruction upon which the straightedge cannot be placed. Form a continuation of the line beyond the obstacle.

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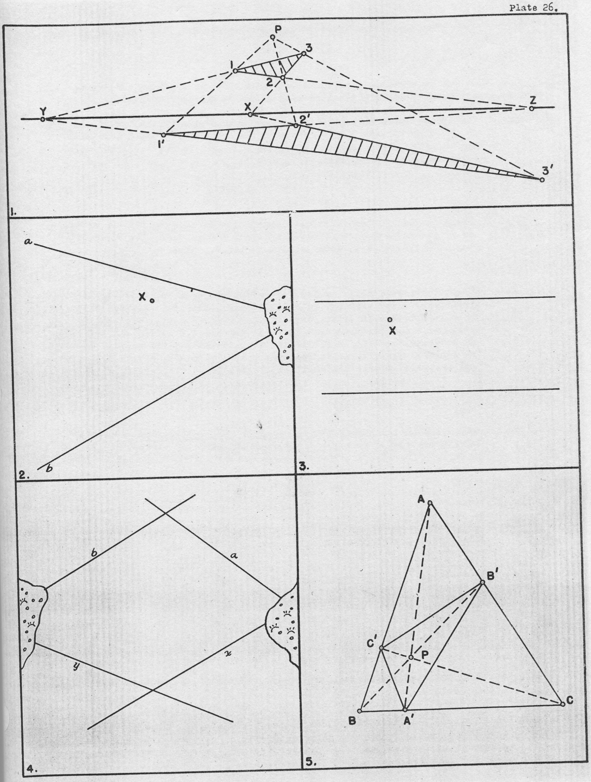
* Corresponding vertices are those which lie collinear with P. Corresponding sides join corresponding vertices.


P
3
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Y
X
Z
2'
1'
3'
1.
a
X
b
2.
X
3.
b
a
y
x
4.
A
B'
C'
P
C
B
A'
5.

## THE STRAIGHTEDGE

We introduce here one of the most far-reaching and beautiful theorems in geometry - a theorem discovered by Pascal at the age of 16 and published in 1640. It is given without proof:

FIG. 1. Six arbitrary points are selected upon a conic and numbered arbitrarily 1, 2, 3, 1', 2', 3'. Lines are drawn from 1 to 2' and 3'; from 2 to 1' and 3'; from 3 to 1' and 2'. The points X, Y, Z of intersection of 2,3' and 2',3, etc., lie on a line. The converse of the theorem also holds. State it.

Interchange two of the numbers in Fig. 1 and construct the new Pascal line. By renumbering in all ways six such given points we may obtain 60 Pascal lines. For a study of this set of lines see W. H. Bunch, Am. Math. Monthly, Vol. XL, p. 251, 1933.

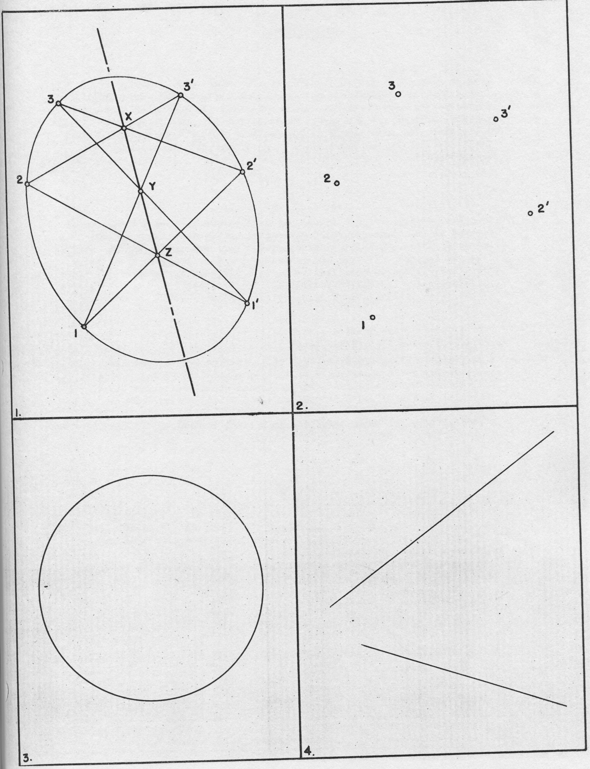
FIG. 2. A conic may be drawn through any five given points, no three of which are collinear. Construct one further point upon the conic whichpasses through the five points of Fig. 2. (Hint: Using Pascal's theorem, draw any line L through 2 which is to contain the desired point 1'. This line L meets 1,2' in Z. The lines 2,3' and 2',3 meet in X. XZ, the Pascal line, meets 1,3' in Y. The line 3,Y cuts L in 1'). Other points of the conic are located by varying the chosen line. Thus by means of the straightedge alone we are able to construct a conic "pointwise."

FIG. 3. The given circle is a special conic upon which the theorem of Pascal must necessarily hold. Choose a set of six points at random upon this circle and construct a Pascal line of the set. (The converse of Pascal's theorem does not hold for the circle).

FIG. 4. A conic may degenerate into a pair of lines. Select three points upon each line of Fig. 4 and establish a Pascal line.

Consider the construction of Fig. 4 if five points are selected upon one line and the sixth on the other.

It is said that Pascal deduced over 400 corollaries from his conic theorem. Give some historical notes:


3
3'
X
2
2'
Y
Z
1
1'
1.
3
3'
2
2'
1
2.
3.
4.

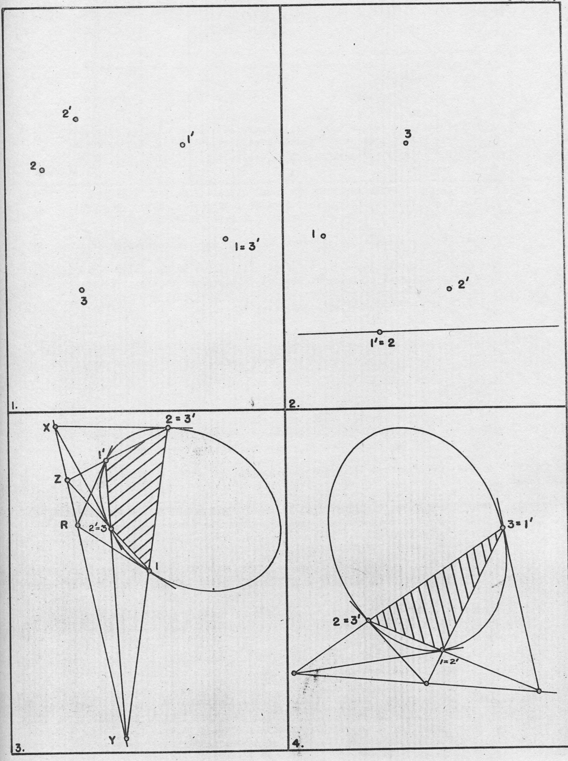
## THE STRAIGHTEDGE

FIG. 1. Given five points 1, 2, 3, 1', 2', no three of which are collinear. Construct the tangent at any one of these points to the conic determined by the five points. (Hint: Suppose that the missing point 3' has merged with point 1. In so doing, the line 1,3' approaches the position of a tangent to the conic. Therefore, establish the Pascal line by finding two points on it - the intersection of (1,2'; 1',2) and of (2,3'; 2',3). Now the line 1',3 cuts this Pascal line in a point Y. Through Y also passes 1,3', the desired tangent).

FIG. 2. Given four points 1,2 = 1',3,2' and a tangent to the circumconic at one of these points. Construct one (and consequently "all") further points on the conic. (Hint: The intersection of 1,2' and 1',2 is Z, a point of the Pascal line. Draw any line through 1 which is to meet the conic in 3'. This line is cut by 1',3 in Y, a second point of the Pascal line. Line 2',3 cuts the Pascal line in X and since 2,3' passes through this same point, the intersection of 1,3' and 1',3 is determined.

FIG. 3. A degenerate case of Pascal's theorem leads to an interesting theorem on inscribed quadrilaterals. Of six points on a conic, let 3' merge with 2 and let 3 merge with 2'. The Pascal line then is determined by the intersections of the three pairs of lines: (1',2;1,2'), (1,3';1',3), and (2,3';2',3), the last pair being tangents to the conic. Thus the theorem:
THE OPPOSITE SIDES OF A QUADRILATERAL INSCRIBED IN A CONIC TOGETHER WITH THE PAIRS OF TANGENTS AT OPPOSITE VERTICES MEET IN FOUR COLLINEAR POINTS.

FIG. 4. One further degenerate case produces a theorem on inscribed triangles. Let the six points on a conic merge together in three pairs: 1'=3, 1=2', 2=3'. Then lines 2,3'; 1,2'; and 1',3 are tangents to the conic. The Pascal line is determined in the usual way, bringing to light the theorem:
IF A TRIANGLE IS INSCRIBED IN A CONIC, THE TANGENTS AT THE VERTICES MEET THE OPPOSITE SIDES IN THREE COLLINEAR POINTS.


2'
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3
1.
3
1
2'
1' = 2
2.
X
2 = 3'
1'
Z
R
2' = 3
1
3.
Y
3 = 1'
2 = 3'
1 = 2'
4.

## THE STRAIGHTEDGE

FIG. 1. Number differently each set of points 1, 2, 3, 4, so that one set is not a cyclic change of another. Join these points in succession. Each figure now represents a simple quadrilateral.

FIG. 2. The four points 1, 2, 3, 4 are given. Six lines and three further points are formed by joining them in all possible ways. This configuration is called a complete quadrilateral. The points 1, 2, 3, 4 are its vertices; the lines 1, 2; 1,4; 2,3; 3,4; 1,3; 2,4 are its sides; and X, Y, Z are its diagonal points.

FIG. 3. IF FIVE PAIRS OF CORRESPONDING SIDES OF TWO QUADRILATERALS (1,2; 1'2'), (3,4; 3'4'), (1,3; 1',3'), (2,3; 2',3'), (1,4; 1'4') INTERSECT IN POINTS A', A, C, B, D' ON A LINE, THEN THE SIXTH PAIR (2, 4; 2', 4') INTERSECTS IN D, A POINT OF THE SAME LINE.

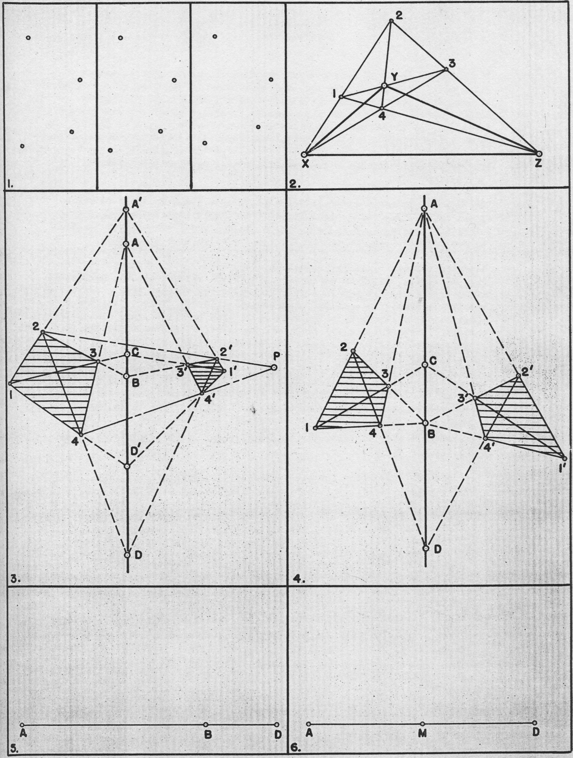
PROOF: Consider triangles 123 and 1'2'3'. Since their corresponding sides intersect in three points A',B,C on a line then by Desargues theorem they are in perspective from some point P. For the same reason, P is the center of perspective for the other three triangles of each quadrilateral, all having the same axis of perspective. Now since two corresponding sides of triangles 124 and 1'2'4' meet in A' and D' their third corresponding sides (2,4; 2', 4') intersect in D, a point of the same line.

FIG. 4. In Fig. 3 let A' approach A and D' approach B. The preceding argument and theorem would in no way be affected. Fig. 3 then reduces to the pair of quadrilaterals shown where the line AD on which the five pairs of corresponding sides intersect is the diagonal line AB (see Fig. 2) of either quadrilateral. By the foregoing theorem, IF THE COLLINEAR POINTS A, B, C BE SELECTED THEN POINT D IS UNIQUELY DETERMINED AS A POINT OF THEIR LINE AND IS INDEPENDENT OF THE "SUPERSTRUCTURE" CONSTRUCTION.

These four collinear points so related are said to be harmonic, the pairing indicated by the symbol; (AB; CD). We say that each pair is conjugate with respect to the other. Given any three of the points, the fourth harmonic point may be located by means of the straightedge alone. Sets of harmonic points have important meaning in metrical geometry.

FIG. 5. Given the three points A, B, D on a line. Locate C so that (AB;CD) is harmonic. (Hint: The location of the point C is independent of the particular superstructure used. Thus draw two arbitrary lines from A meeting a line drawn arbitrarily from D. Then draw two lines from B forming a quadrilateral with the line through D as a side. The sixth side of this complete quadrilateral produces the point C).

FIG. 6. Given that M is the midpoint of the segment AD. Locate N such that (AD;MN) is harmonic. Discuss carefully from the metric viewpoint.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D'
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1'
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4.
A
B
D
5.
A
M
D
6.

THE STRAIGHTEDGE

FIG. 1. If the harmonic set (AB;CD) and its complete quadrilateral superstructure be projected from a point P in space upon a plane $\pi$ a new quadrilateral and corresponding harmonic set of points A',B',C',D' are formed. It is thus evident that FOUR HARMONIC POINTS REMAIN HARMONIC WHEN PROJECTED. (A point of projection on the line of harmonic points is excluded). For example, the harmonic points (AB;CD) may be projected from point 1 onto the line 2,4, the points A,B,C,D projecting into 2,4,X, and D, respectively. Thus the set (2,4;XD) is harmonic.

Lines joining a point and each of four harmonic points are themselves called harmonic.

FIG. 2. Draw two secant lines from any selected point X to the given conic cutting out the quadrilateral 1,2,3,4. This complete quadrilateral has the diagonal points X,Y,Z. <u>Each side of this diagonal triangle X,Y,Z, is the polar of its opposite vertex with respect to the conic.</u> A vertex is called a <u>pole</u>. The triangle XYZ is said to be <u>self-polar</u> to the conic.

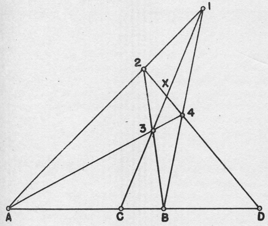
From Pascal's theorem on inscribed quadrilaterals, the tangents at 2 and 4 meet on the line YZ at the point L. Now (KM;YZ) are harmonic and by Fig. 1. so is the set (KX;2,4)harmonic. Thus by drawing a single secant 2,4 through X we may determine K so that (KX;2,4) is harmonic. The point L is determined by the tangents at 2 and 4. Thus the polar KL of X is located without recourse to the second secant 1,3. This latter line may be drawn entirely at random. On the other hand, any one of the second secants may be drawn through X in order to locate the polar and the first secant could be drawn arbitrarily.

FIG. 3. Because of the preceding argument: THE POLAR OF P IS THE LOCUS OF THE INTERSECTION OF THE CROSS-JOINS OF POINTS WHERE SECANTS THROUGH P CUT THE CURVE. It should be noticed from the remarks under Fig. 2. that if the arbitrary line P,1,2 cuts the polar in Q <u>then (PQ;1,2) is a harmonic set of points.</u> State this in other words:

FIG. 4. Discuss Fig. 3. as the secant through P approaches the position of a tangent.

Now construct the tangent from P to the curve in Fig. 4. This straightedge construction of a tangent to a conic from an external point is remarkably simple and noteworthy.

FIG. 5. In Fig. 3. we saw that a secant through the pole cuts the curve and the polar line in three points harmonic with the pole. If we select any point Q on the given line <u>p</u> of Fig. 5. and form its polar then this polar line must pass through the pole P of <u>p</u>. In other words, IF Q LIES ON THE POLAR OF P THEN P LIES ON THE POLAR OF Q. From this construct the pole of the line <u>p</u>.

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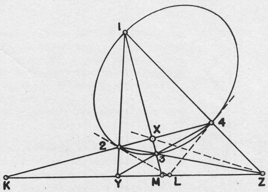


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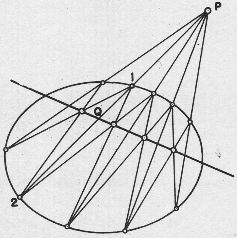


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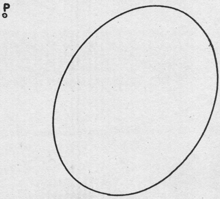


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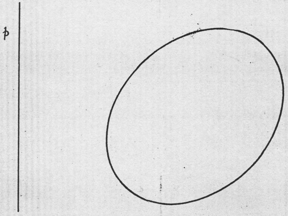



5.

FIG. 1. For each point P in the plane of a fixed conic there corresponds a line, its polar p; and to the line p corresponds a definite point P. To a range of points corresponds a pencil of polar lines all passing through the pole of the carrier. This is a reciprocal affair between points and lines which forms the famous principle of duality first conceived by Poncelet in a Russian prison and used by Brianchon to transcribe the theorem of Pascal into the one that bears his name. Unfortunately perhaps, this transcription took place over 150 years after Pascal.

FIG. 2. After Brianchon, let us transcribe the theorem of Pascal into poles and polars. Each of the six points on the conic of the Pascal configuration has for polar the tangent line at that point. Thus we produce a six-sided figure circumscribing the conic, each vertex of which is a pole of a side of Pascal's incribed figure. The point of intersection of two corresponding sides of Pascal's figure transcribes into its polar line; that is, the join of two corresponding vertices of the circumscribed figure. These three points lie on the Pascal line. Their polars accordingly pass through a point - the pole of the Pascal line. We have thus established the theoren of Brianchon: IF A HEXAGON CIRCUMSCRIBES A CONIC THE THREE LINES JOINING OPPOSITE VERTICES PASS THROUGH A POINT. Draw the figure, using different colors for the two reciprocal theorems of Pascal and Brianchon.

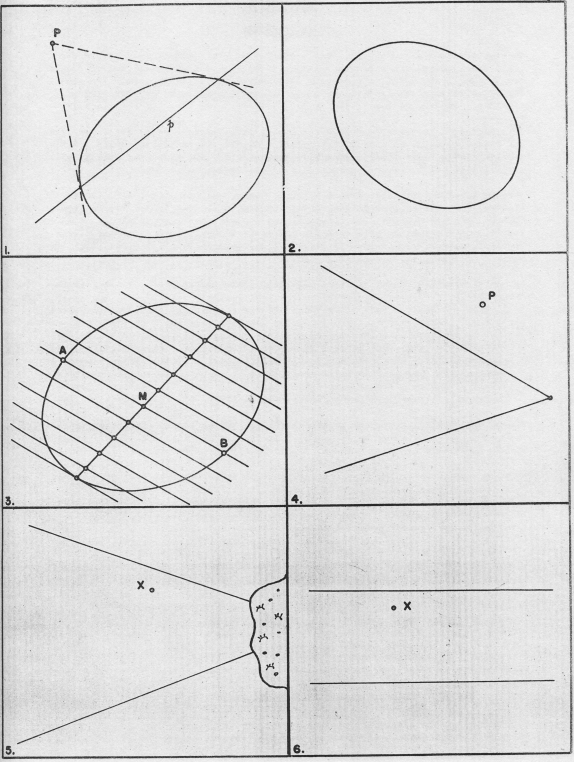
FIG. 3. We define a diameter of a conic as the polar of an infinitely distant point. In Plate 29,6 we saw that the infinitely distant point on the extension of the segment AB formed with the midpoint M of AB a pair of points harmonic with A,B. Therefore, a diameter is the locus of midpoints of a set of parallel chords of the conic. Any two non-parallel diameters intersect in the center of the conic. What is the polar line of the center?

FIG. 4. A conic may degenerate into a pair of lines as shown. The polar of a point P with respect to such a special conic passes through the intersection of the two lines. Why?

Draw the figure.

FIG. 5. Using the ideas presented here, solve again and in a different manner (See Plate 26,2) the problem of drawing a line through X and the inaccessible intersection of two given lines.

FIG. 6. Draw a line through X parallel to the two given parallel lines.



1.
P
p
2.
3.
A
M
B
4.
P
5.
X
6.
X

## SECTION VI

## LINE MOTION LINKAGES

The most prominent motion is circular. The conversion of the easily attained circular motion into motion along a straight line is of prime importance to the engineer and mechanic.* This was especially true 75 years ago when modern machinery was in its formative stage. Steam had recently been applied both to land and water vehicles but poorly riveted boilers and clumsy levers improperly lubricated played havoc with life and limb.

The generation of line motion was no doubt of concern to mathematicians since the time of Archimedes and, because no solution was apparent, many confused the problem with squaring the circle. A solution was first given by Sarrus in 1853, another by Peaucellier in 1864, both of which lay unnoticed until Lipkin, a student of Tschebyscheff, independently recreated Peaucellier's mechanism.

Fanned by Sylvester's enthusiasm, interest in general linkwork immediately flamed high to attract the attention of men like Cayley, Kempe, Hart, Darboux, Clifford, Koenigs, Sir William Thompson, Darwin, Mannheim, and a host of lesser minds. The epidemic was so fierce and so universal that the subject was drained almost completely dry in the short span of five or six years. The drop in interest followed Sylvester's departure for America and Kempe's proof of the remarkable theorem that any algebraic curve, no matter how complex, can be described by a linkage.

Evidence of Sylvester's somewhat justifiable enthusiasm is the following quotation from his Collected Work, III:

> "It would be difficult to quote any other discovery which opens out such vast and varied horizons as this of Peaucellier - in one direction, as has been shown, descending to the wants of the workshop, the simplification of the steam engine, the revolutionizing of the millright's trade, the amelioration of garden-pumps, and other domestic conveniences (the sun of science glorifies all it shines upon), and in the other soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unsuspected quarter on the researches of such men as Abel, Riemann, Clebsch, Grassman, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth."

Although the drawings given here seem to indicate otherwise, there is no necessity that bars or links be straight. Indeed, this would beg the question. The line joining two joints is the effective distance and the only requirement is that all bars be plane, inextensible members.

In making models of the various linkages, the student should obtain colored cardboard (poster board) about 12-ply; an eyelet punch; and boxes of #2 and #3 eyelets. Use the #2 eyelet to join two links; #3 to join three or four links. Cut the cardboard into strips about one-half inch wide with a phototrimmer and mount the model on a cardboard background. To insure greater accuracy, two bars of the same length should be punched simultaneously.

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* The reverse problem of converting line motion into circular motion is a simple one.

# THE PEAUCELLIER CELL

FIG. 1 and 2. The Spear Head and Kite are formed of jointed bars which are equal in pairs. Let

$$OA = OB = a; \qquad AP = BQ = PB = AQ = b.$$

FIG. 3. The Spear Head and Kite may be combined to give the six-bar linkage of Fig. 3 in which the joints O, P, Q are always collinear. Let M be the midpoint of PQ. Then

$$(OM)^2 + (AM)^2 = a^2; \qquad (PM)^2 + (AM)^2 = b^2;$$

which give, by subtraction:

$$(OM)^2 - (PM)^2 = a^2 - b^2.$$

The left member may be written as the product:

$$(OM - PM)(OM + PM) = k^2,$$

where $k^2 = a^2 - b^2$, $(a > b)$. Since $PM = MQ$, this last equation may be rewritten as:

$$(OP)(OQ) = k^2.$$

This, as may be recalled (see Plate 16.2), is the fundamental principle of inversion. With this mechanism then we may obtain the inverse of any given curve - where the circle of inversion has radius k, i.e., the distance from O to P, when P and Q are coincident.

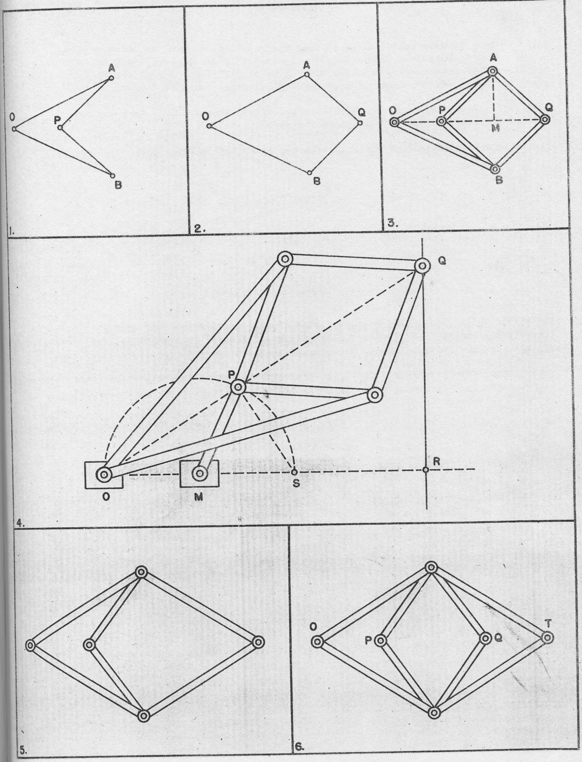
FIG. 4. In particular, we may obtain line motion by inverting a circle which passes through the origin. By fixing O and attaching a seventh bar, MP, as shown so that P describes a circle through O, the point Q traverses a straight line. To see this in an elementary fashion, let the linkage be placed in an arbitrary position as indicated. Draw a line through Q perpendicular to the line OM of fixed points. It is evident that the right triangles OSP and ORQ are similar; thus

$$OP/OR = OS/OQ \qquad \text{or} \qquad (OR)(OS) = (OP)(OQ).$$

But (OP)(OQ) is constant and so therefore is (OR)(OS). Since S is a fixed point and this product is constant, then R is accordingly fixed and the point Q lies always on the perpendicular at R. This is the celebrated discovery of Peaucellier in 1864.

FIG. 5. presents the negative Peaucellier cell. Derive the fundamental relation for this arrangement and attach an extra bar for line motion. Explain.

FIG. 6. This is the symmetrical double Peaucellier cell formed of either two kites or two spear heads. What combination of the points O, P, Q, T gives the inverse property?


1.
O
P
A
B
2.
O
A
Q
B
3.
O
P
A
Q
M
B
4.
P
Q
O
M
S
R
5.
6.
O
P
Q
T

## THE PEAUCELLIER CELL

FIG. 1. This is the assemblage of a Kite and a Spear Head in reverse positions. The inverse property is preserved if we extend RO and SO equal lengths to A and B and then add the equal bars AP, BP so that OA and AP are proportional to OR and RQ. Since $\angle$ RQS = $\angle$ APB; $\angle$ AOB = $\angle$ ROS, this arrangement has the inverse property and the product of the distances OP and OQ is constant.

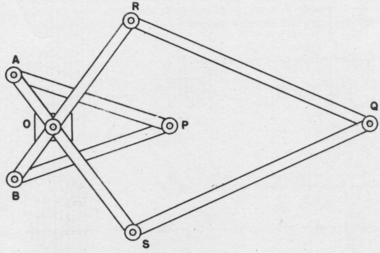
If O is fixed and P be made to move on a circle through O, then Q describes a line. The motion here is a much faster one than given by the mechanism of Plate 32,4.

Find the value of the constant (OP)(OQ) if the spear head is half the size of the kite.

FIG. 2. Let two proportional kites be arranged as shown so that AP/PR = RO'/O'S with AP = AT; PR = RO' = RT; O'S = ST. Let $\angle$ TAP = $2\theta$ = $\angle$ TRO' and $\angle$ APR = $\phi$ = $\angle$ ATR = $\angle$ RO'S. Then $\angle$ PRO' = $2\pi - 2\phi - 4\theta$. Since triangle PO'R is isosceles, $\angle$ RPO' = $\phi + 2\theta - \pi/2$ and therefore $\angle$ APO' = $\pi/2 - 2\theta$. Thus, since $\angle$ PAT = $2\theta$, PAM is a right triangle with the line joining P and O' always perpendicular to the bar AT. Accordingly, if we fix O' and move AT parallel to itself then P will describe a line perpendicular to AT. To do this, attach the bar OA equal in length and parallel to O'S and fix the point O.

What is the path described by any selected point of PR? By a point Q on PR extended such that PR = RQ?

Remove the bar OA, free the point O' from the plane, and fix P. Then attach one end of a bar to T which is equal in length to AP. Fix its other end to the plane at Q such that PQ = AT. This arrangement permits O' to move on a line perpendicular to PQ. Establish this fact.

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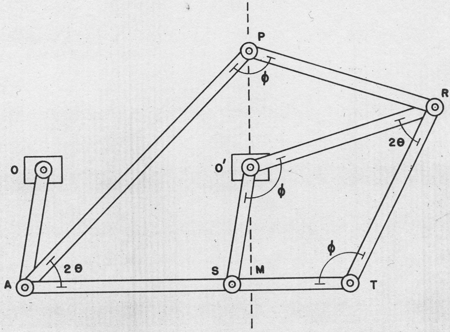



2.

# THE PEAUCELLIER CELL

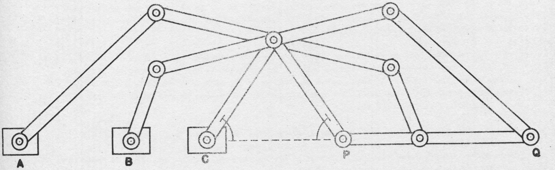
Each of the linkages given here employs a double kite arrangement. There are four kites, the larger equal ones proportional to the smaller ones. In the first the bar PQ moves in line with the three collinear fixed points A, B, C. In the second, PQ moves always parallel to the line of fixed points. Establish these facts for the two mechanisms.

FIG. 1. (Hint: Prove B, C, and P collinear.)

List some possible applications of this linkage.

FIG. 2.

List some possible applications of this linkage.

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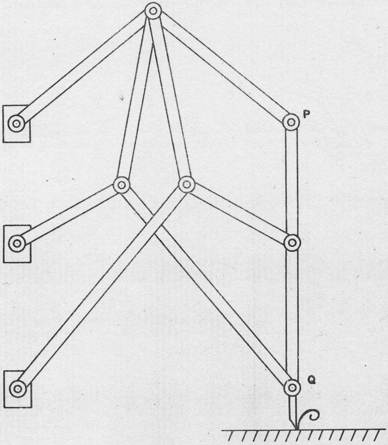



2.

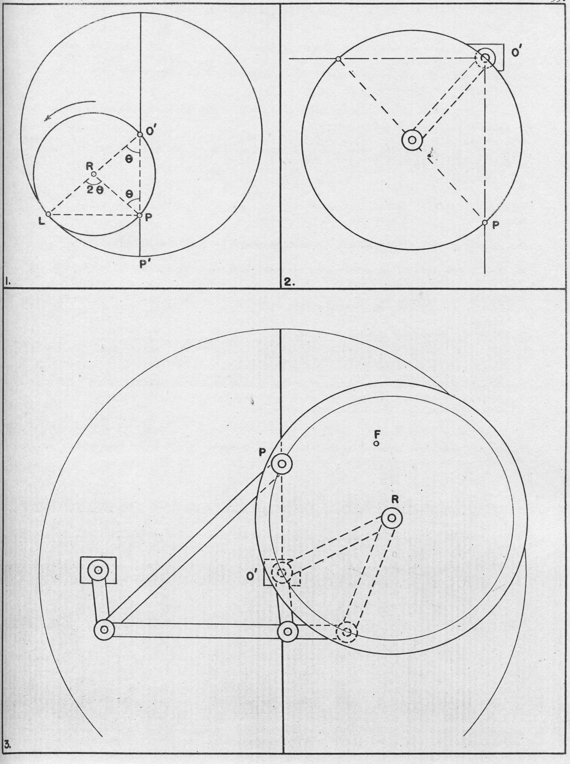
FIG. 1. Consider a circular disk whose diameter equals the radius of a fixed circle within which the disk rolls. Draw the lines O'P' cutting the disk at P and let $\angle$ LO'P = $\theta$. Then arc LP' = $a\theta$, where a is the radius of the circle. Now LO'P is a right triangle with LR = RO' = RP = $a/2$. Thus arc LP = $a\theta$ and evidently this is the position of the disk after rolling on the larger circle through the arc length LP', with the original position of P at P'. Thus the point P, fixed on the rim of the disk, travels along the diameter P'T. Since P is any point on the rim, every point travels on a diameter of the larger circle. The line segment path may be thought of as the two-cusped member of the Hypocycloid family.

FIG. 2. Presented here is a different scheme for the same motion as that of Fig. 1. To the center of the disk (and underneath) is attached a bar equal in length to the radius with its other end fixed to the plane at O'. If some point P on the rim of the disk is moved along a line through O' then every point on the rim moves on a line through O'. The action is just as if the disk were rolling inside a circle twice its size.

FIG. 3. Adapting the ideas of Figs 1 and 2 to the mechanism of Plate 33,2, the bar PR of 33,2 is replaced by the disk having PR for radius. As the linkage is deformed, P travels on a line through O'. By the above principles, every other point on the rim does likewise and the disk moves as if it were rolling within the imaginary circle.

What is the path of any point F of the disk? (Recall the trammel of Archimedes).

Construct some sort of mechanism, following Fig. 1. to give the three-cusped Deltoid; the four-cusped Astroid; with disk radius one-third and one-fourth, respectively, of the radius of the larger circle.


O'
θ
R
2θ
θ
L
P
P'
1.
O'
P
2.
P
F
R
O'
3.

## HART'S LINE MOTION

FIG. 1. Two sets of equal bars $AC = BD$, $PC = PD$ are joined as shown. The points A and B are fixed to a base plane. If we move P so that the angles at C and D are always equal, then triangles APC and BPD are congruent and AP always equals BP. This requires that P lie always on the perpendicular bisector of the segment AB.

It would seem difficult indeed to arrange mechanically for the angles at C and D to be always equal. Surprisingly enough, such is not the case.
For,
Let $AC = BD = a$, $PC = PD = b$. Then select two points R and S on the bars AC and BD respectively, so that

$$RC = SD = b^2/a.$$

Then $$RC/PC = (b^2/a)/b = b/a = PC/AC.$$

Thus $\angle PAC = \angle RPC = \angle SPD = \angle PBD = x$. Furthermore, $\angle APR = \angle PRC = \angle BPS = y$.

Now since $PR = PS$; $PA = PB$; and $\angle RPS = x + y + z = \angle APB$, then triangles APB and RPS are similar.

Accordingly,

$$PR/PA = RS/AB = RC/PC = b/a.$$

Thus if we take the constant distance $AB = c$, then

$$RS = bc/a.$$

That is, if P describes the bisecting line of AB then the distance between the moving points R and S is constant. Conversely, the angles at C and D will remain equal and the point P will describe a line if R and S be joined by a bar of the proper length.

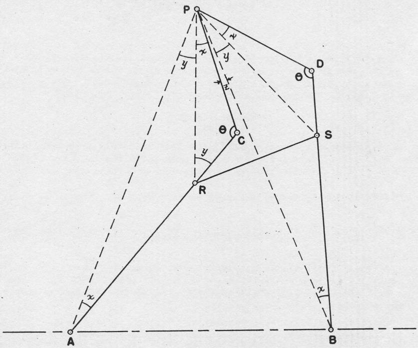
FIG. 2. In building the linkage, take the five links: $AC = BD = a$; $PC = PD = RS = b$; attaching the link RS to R and S such that the distance $RC = DS = c$, where

$$b^2 = ac.$$

(It will be found convenient to take these distances as 2, 4, and 8 inches)

Before attaching the linkage to a base, lay it open so that P is at the uppermost point. The mechanism then forms the letter "A". In this extreme position fix the points A and B along any desired line.

What is the path of any point of the bar PD?

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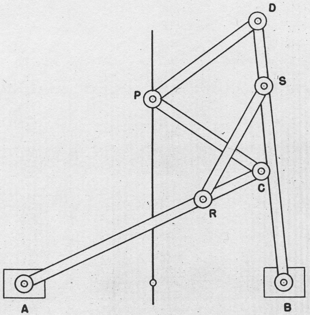



2.

THE HART CELL INVERSOR

FIG. 1. If any four points, O, P, Q, R be selected on the bars of the Hart cell in a line parallel to BD and AC, they will remain in a line as the cell is deformed. Draw the circle through A, P, and Q. Since its center is on the perpendicular bisector of PQ, the line of symmetry of A and C, then C also lies on the circle. Let the circle cut the bar AD in the point T. This point is a fixed point of the bar AD. For, by the secant property of the circle, (Plate 6,1):

$$(DT)(DA) = (DP)(DC),$$

in which the right member is a constant since D, P, and C are fixed points of the bar. In the left member, DA is constant and thus T must be a fixed point of AD. That is, throughout all deformations, A, T, P, Q, C - points fixed on the several bars - are always concyclic.

Since O is a fixed point of AD, we have also by the secant property:

$$(OP)(OQ) = (OT)(OA) = \text{Constant.}$$

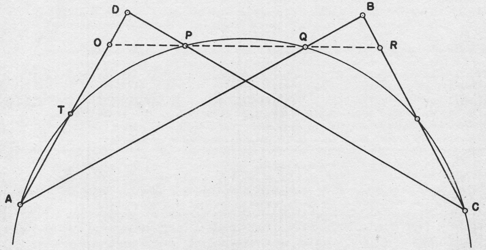
Thus, since the product of the variable distances OP and OQ is constant, this remarkable four-bar mechanism has the same inversive property as that of the Peaucellier cell of Plate 32,3. For line motion then, following the principle of Plate 32,4, we may fix O to the plane and cause Q to move on a circle through O. Thus P describes a line.

FIG. 2. shows the arrangement of the Hart cell for line motion. The extra fifth bar O'Q is attached so that Q travels on a circle through the fixed point O.

What other dispositions of the points O, P, Q, R can be made to produce line motion?

What is the constant value of $(OP)(OQ)$ if $OD = (AD)/2$; if $OD = (AD)/3$?

Compare this cell with the double Peaucellier cell which has twice as many bars.

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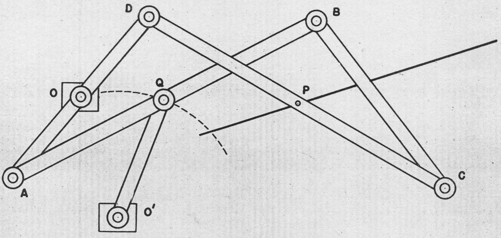



2.

# THE HART CELL: LINE MOTION

The parallelogram of Figure 1 is formed of four bars which are equal in pairs. Use is made of it in the ordinary pantograph, lazy tongs, etc. Outside of that it is comparatively sterile. (But see Plates: 72 & 73) Notice that if a rhombus be formed and flattened, it may be opened at either end. Thus a door which is fastened to its frame by a jointed rhombus may be swung open from either side.

The Hart cell of Figure 2 is the same parallelogram in its crossed position. It is rather remarkable that in this position, the configuration is endowed with unusual and surprising properties. It will be noticed that throughout all deformations, the angles at A and C are equal to each other while those at D and B are also equal.

In Figure 3 one end of the Kite is attached to the plane. If the two equal bars AH and AK be made to rotate about A in opposite directions at equal rates, then obviously, the point P will travel along a straight line through A. By means of the Hart cell, this can be accomplished.

Figure 4 shows the union of two contra-parallelograms, the short bar of the larger cell acting as the long bar of the smaller one. Notice that the angles at E, D, and B are always equal to each other as the linkage is deformed. If the points A and D are attached to the plane, then as the bar AB rotates about A in one direction, the bar EA rotates in the opposite direction. If it is possible to make AB and AE rotate at equal rates, then by combining the kite mechanism of Fig. 3 with this, we will have a linkage for line motion. If we demand that angle EAD be always equal to angle DAB, then the two cells must be similar. This means, of course, that their corresponding sides should be proportional.

Thus

$$AE/AD = AD/AB \qquad \text{or} \qquad (AD)^2 = (AE)(AB),$$

For angle EAD to equal angle DAB.

Figure 5 shows the linkage built as the combination of Fig. 3 and Fig. 4. The bar AD of Fig. 4 has been removed and the two points, A and D, are attached at the proper distance (AD = EF = BC) to the plane.

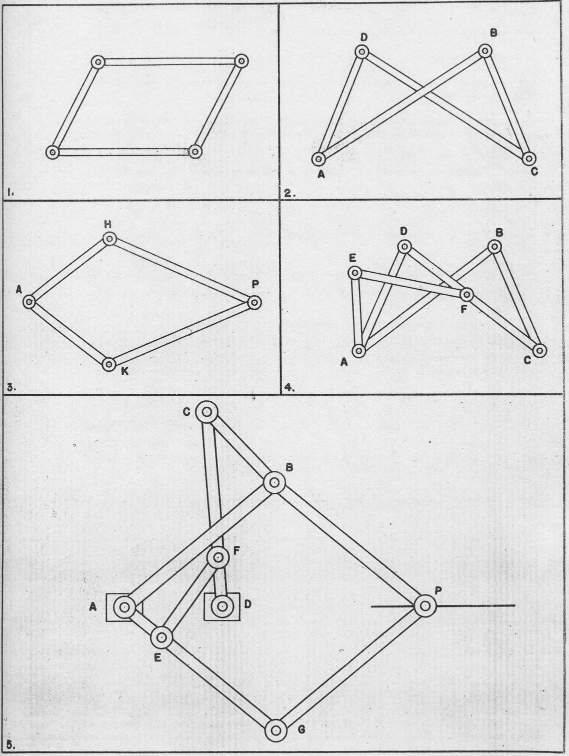
In constructing the model of Figure 5, it will be found convenient to take the following lengths:

$$AE = 2 = FD$$

$$AD = 4 = EF = BC$$

$$AB = 8 = AG = CD$$

Plate 38.



As shown in the figure, we select the following lengths:

$AB = BC = CD = CP = 4a; \qquad AD = DC' = C'D' = C'P = 2a; \qquad AD' = a.$

The points A, D', and B are attached in a straight line to the base plane. We shall show that P lies always on this line.

From the selected lengths, quadrilaterals ABCD and ADC'D' are similar since they contain a common angle. Thus the angles of the first, x, y, z, are equal to those of the second at corresponding vertices.

Moreover, $\angle ADC' = x$. Then $\angle C'DC = z - x = \angle C'PC$ by virtue of the spearhead PCDC'.

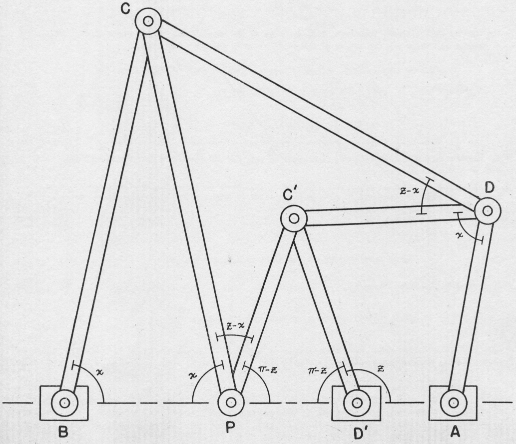
But, $\qquad \angle AD'C' = z \quad$ and $\quad \angle C'D'P = \pi - z = \angle C'PD'$.

Accordingly, $\qquad \angle CPD' = (z - x) + (\pi - z) = \pi - x$

and therefore the points B, P, and D' are collinear. Consequently, P must move on the straight line AD'B.

1. What is the path of a point Q on the extension of PC such that CQ = PC?

2. Replace the bar CP by a circular disk of radius CP having its center at the movable point C. What is the path of any point of the disk? Of the rim of the disk?


C
C′
D
z - x
x
z - x
x
π - z
π - z
z
x
B
P
D′
A

A KEMPE LINE MOTION

FIG. 1. We combine two similar quadrilaterals ABCD and ADC"D', whose angles are

$$\angle ABC = x = \angle ADC''$$

$$\angle ADC = y = \angle AD'C''.$$

Select the following lengths:

$$AB = BC = CD = 4a; \qquad DA = DC'' = C''D' = 2a; \qquad AD' = a,$$

the smaller quadrilateral thus being half the size of the larger one. In the quadrilateral ABCD, we express (in two ways) the length of the diagonal AC by the Law of Cosines:

$$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos x = (DC)^2 + (DA)^2 - 2(DC)(DA)\cos y,$$

which reduces to: $32a^2 - 32a^2\cos x = 20a^2 - 16a^2\cos y,$

or $$2\cos x - \cos y = 3/4. \qquad \text{..........(1)}$$

Now add the bars DC', D'C', and CP, each equal to 2a; and the bar PC' = 4a. Thus CPC'D and DC'D'C" are parallelograms.

Since CP is parallel and equal to DC', it is parallel to D'C" and their projections on the base line, AB, are equal. That is,

$$MR = ND'.$$

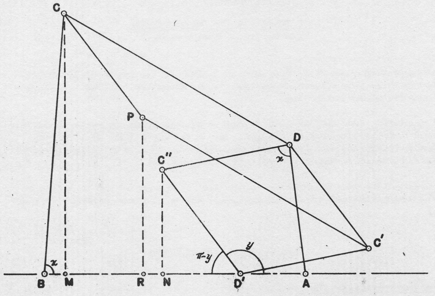
Thus $$BR = BM + MR = BM + ND' = 4a.\cos x + 2a.\cos(\pi - y).$$

By virtue of (1), this becomes:

$$BR = 2a(2\cos x - \cos y) = 3a/2, \quad \text{a constant.}$$

Accordingly, if A, D', and B are fixed on a line, then R is a fixed point and P will describe the perpendicular bisecting line of BD'.

FIG. 2. In building the linkage, the original bars DC" and D'C", which are of no service, may be discarded. It will be found convenient to take a = 2 inches.

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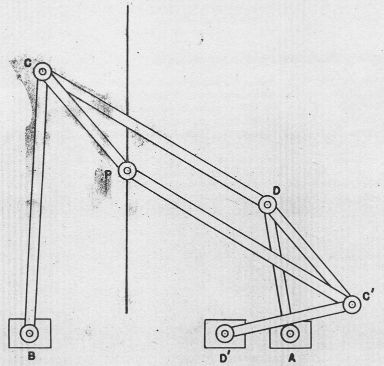


2.

## SECTION VII

## THE STRAIGHTEDGE WITH IMMOVABLE FIGURE

(The Geometry of Poncelet-Steiner)

The constructions of this section are those which can be made with the movable straightedge when given somewhere in the plane a figure already drawn. Such constructions have been of interest to mathematicians for several hundred years.

It should be carefully noted that the system composed of immovable circle and movable straightedge is equivalent to movable straightedge and compasses only if the center of the circle is given. (It has been proved that the center of a circle cannot be located by means of the straightedge alone. See H. Steinhaus, Mathematical Snapshots, New York, 1938, 44).

In order to shorten the labor in complicated constructions, it is suggested that the student omit those preliminary constructions already made which would confuse the picture or obscure the main objective.

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## THE STRAIGHTEDGE WITH IMMOVABLE CIRCLE

FIG. 1. Draw through P a parallel to the given diameter AOB of the fixed circle. (Hint: The construction is exactly similar to that given in Plate 29,6.) Note that the fixed circle here plays only the role of presenting a bisected segment AOB - a concept that will prove of frequent use in later constructions.

FIG. 2. Draw through P a line parallel to the given line L. (Hint: From any point M of L draw MO. From P draw a parallel to MO meeting the circle in two points. Draw the cross diameters from these two points. The lines thus joining the extremities of these diameters are both parallel to MO and cut the line L in a bisected segment.)

FIG. 3. Draw the diameter of the fixed circle that is parallel to the given line L. (Hint: Draw an arbitrary line through O meeting L in A. From any other point B of L, draw a second line parallel to OA cutting the circle in two points. By cross diameters, establish a third line parallel to the first two, thus obtaining a bisected segment on L. The construction is completed according to Plate 29,6.)

FIG. 4. Locate the internal and external centers of similitude of the given fixed circle and the hypocircle* with center O' and passing through the point A'. (Hint: The internal center, I, and the external center, E, lie on the line of centers OO'. Draw the diameter BOA that is parallel to O'A'. Then the lines BA' and AA' determine and required points.)

* The word "hypocircle" is a contraction of "hypothetical circle". It will be frequently used to denote the circle determined by its center and one of its points. Although it cannot be drawn except when the compasses are allowed, the student should indicate it by dotted lines or by its colored interior.

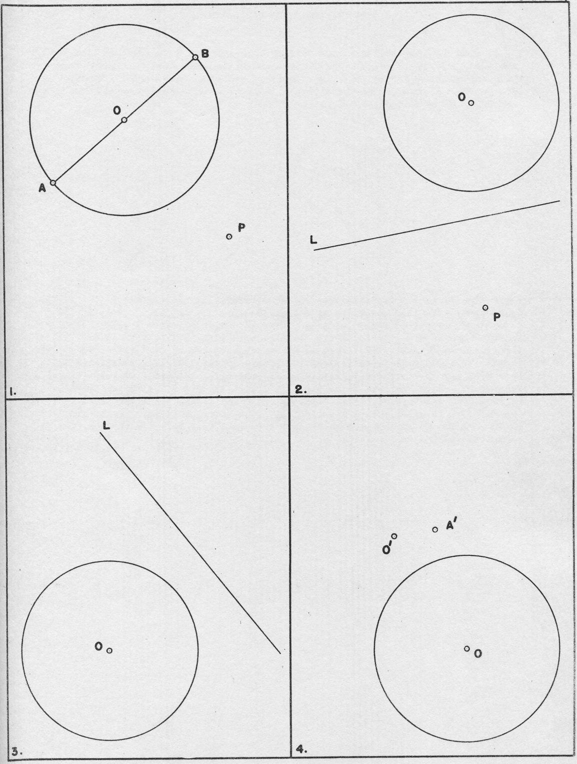
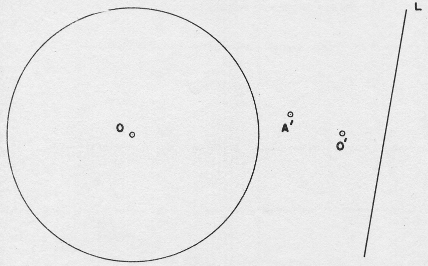
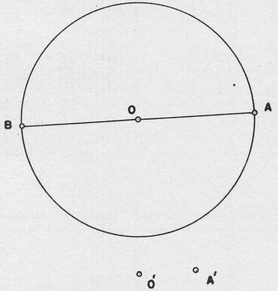

1.
A
B
O
P
2.
O
L
P
3.
L
O
4.
O'
A'
O

FIG. 1. Given the fixed circle with center O. Find the intersections of the given line L with the hypocircle O'(A'). (Hint: Determine the external center of similitude, E, of the fixed circle and the hypocircle. Extend A'O' to meet L in B'. Draw the diameter AOK parallel to A'O'. The line EB' meets AOK in B. Through B draw line M parallel to L which meets the fixed circle in X' and Y'. Lines EX' and EY' cut L in the desired points).

FIG. 2. Find the intersections of the fixed circle with center O and the hypocircle O'(A'). Generally, the points A and A' would not be such that OA and O'A' are parallel. However, in order to shorten the labor of the student, the radii are here given parallel. (Hint: Proceed to locate the radical axis of the two circles and find its intersections with the fixed circle. First locate the center of similitude E. Let EA cut the two circles in C and C' and let EB cut them in D and D'. Then BC is perpendicular to EA and A'D' is perpendicular to EB. Thus quadrilateral BCA'D' is inscribed to a circle since its opposite angles are right angles. The circle drawn about this quadrilateral meets the two given circles in B,C and A',D' and the lines BC and A'D' are therefore its radical axes with the given circles. Since the radical axes of three circles meet in a point (see Plate 6,5), the intersection of BC and A',D' is a point on the radical axis of the two given circles. The line through this point perpendicular to OO' meets the fixed circles in the required points). (See Plate 23,2).




1.




2.

FIG. 1. Bisect the given segment AB. (Hint: Draw the diameter parallel to AB by Plate 41,3.)

FIG. 2. Construct a rectangular network of lines. (Hint: The diagonals of a rhombus are perpendicular.)

FIG. 3. Bisect the given angle ABC. (Hint: Draw diameters DOF and GOH parallel to AB and BC, respectively. The line FG is parallel to the desired bisector. Why?)

FIG. 4. Erect a perpendicular to the given line at P. (Hint: Draw an arbitrary chord AB of the given circle parallel to the given line. Draw the diameter AOC. Then BC is parallel to the desired perpendicular at P.)

FIG. 5. Transfer the distance O'A' onto the given line L. (Hint: Locate the center of similitude E of the given circle and the hypocircle O'(A'). Then draw the parallel to L through O.)

FIG. 6. Find the center of the circle which passes through A, B, and C. (Hint: The circumcenter of ABC is the intersection of the perpendicular bisectors of the sides of the triangle ABC.)

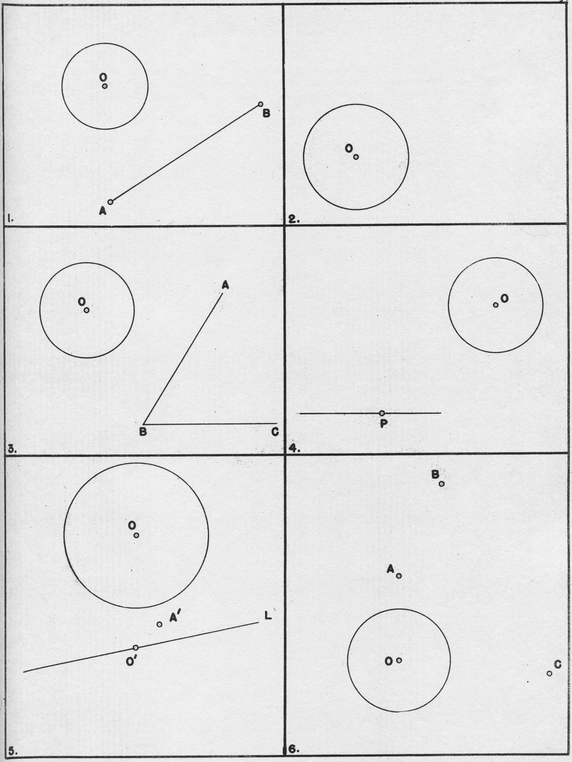

1.
O
A
B
2.
O
3.
O
A
B
C
4.
O
P
5.
O
A′
L
O′
6.
B
A
O
C

FIG. Find the intersections of the two hypocircles C(A) and C'(A'). (Hint: Proceed to establish their radical axis. First transfer the distance C'A' to a line parallel to CA. Then locate the external center of similitude E of the two hypocircles. Complete the construction according to Plate 23,2.)

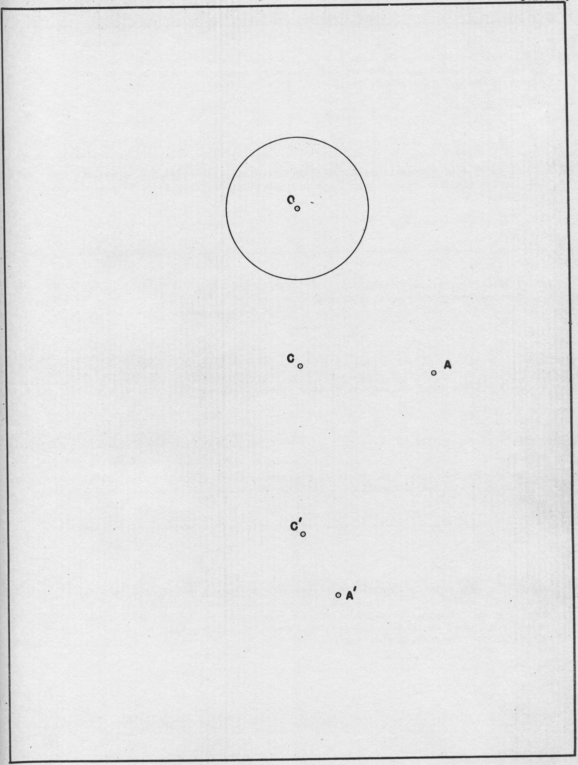

O
C
A
C′
A′

FIG. 1. Draw the line through the corner C of the given square parallel to the diagonal BD. (Hint: The points B, D, and the center of the square form a bisected segment parallel to the desired line. This is a direct application of Plate 29,6.)

FIG. 2. Establish the midpoint of the side BC of the given square. (Hint: Extend DC to meet an arbitrary line through B. Then construct the polar of this point with respect to the two parallel lines AD and BC.)

FIG. 3. Construct a rectangular network of lines. (Hint: Extend the sides of the given square; then draw lines through vertices and midpoints of sides.)

FIG. 4. Draw a line through the center of the given square parallel to a side.

FIG. 5. Draw the line through P parallel to the given line L. (Hint: Join P with the center O of the given square. Then construct parallels to PO through the vertices B, D. These lines cut L in a bisected segment.)

FIG. 6. Draw the line through P parallel to the given line L. (Hint: A neat solution is afforded by the theorem of Desargues. Obtain triangles in perspective as follows: produce A'D and A'D' to meet L in B' and C' respectively. Select an arbitrary point R on the diagonal AA'. Let the point of intersection of B'R and AD' be B; that of RC' and AD be C. Then BC is parallel to L. Why? The construction is completed by Plate 31,6.)

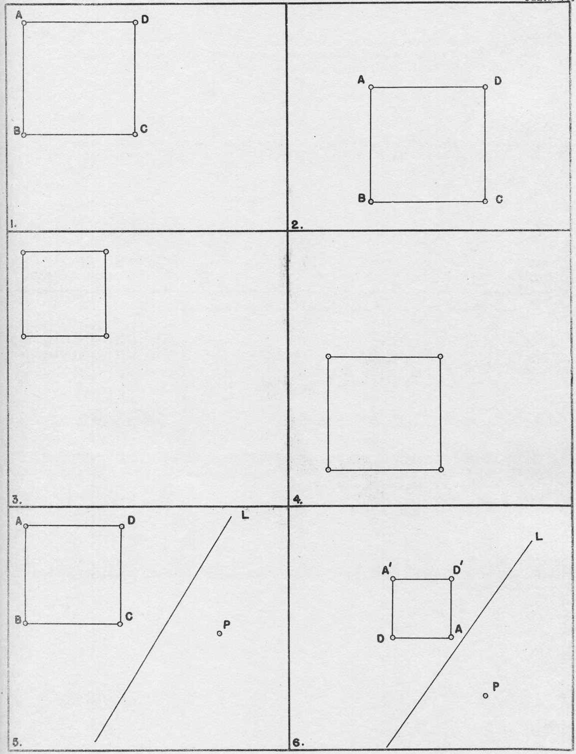

A
D
B
C
1.
A
D
B
C
2.
3.
4.
A
D
L
B
C
P
5.
L
A'
D'
D
A
P
6.

FIG. 1. The line XY is drawn through the center O of the given square. Draw the perpendicular to XY through O. (Hint: Through Y draw the parallel to BD, meeting BC in U. The line through U parallel to DC meets AD in W. The line WO is perpendicular to XY. Why?)

FIG. 2. Draw the perpendicular from P to the given line L. (Hint: Draw through the center of the square the line parallel to L.)

FIG. 3. Discuss the possibility of finding the centroid of triangle ABC.

FIG. 4. Discuss the possibility of finding the orthocenter of triangle ABC.

FIG. 5. Discuss the possibility of finding the circumcenter of triangle ABC.

FIG. 6. Reflect the point P in the given line L. (Hint: The reflected point P' lies on the perpendicular from P to L. The segment through the center of the square parallel to PP' offers a bisected length that can be projected onto PP', thus determining P'.)

FIG. 7. Multiply the given angle $\theta$. (Hint: Reflect an arbitrary point of one side of the angle in the other side.)

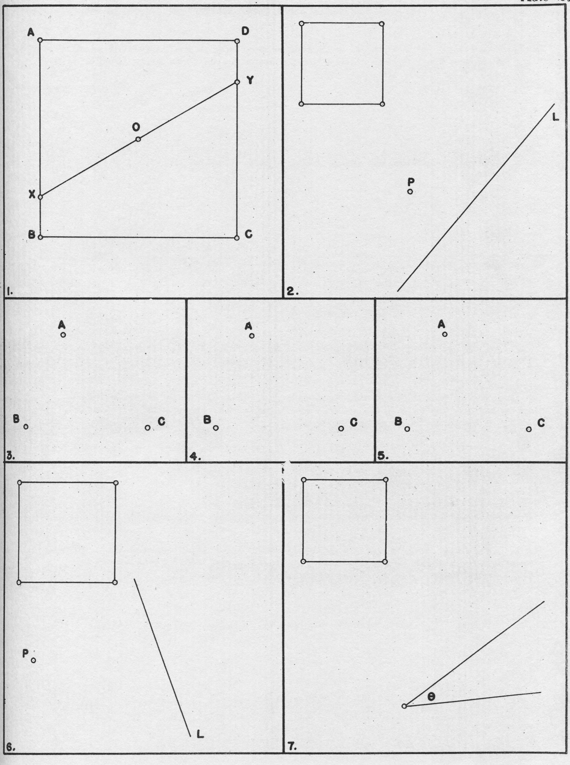

A
D
Y
O
X
B
C
1.
L
P
2.
A
B
C
3.
A
B
C
4.
A
B
C
5.
P
L
6.
θ
7.

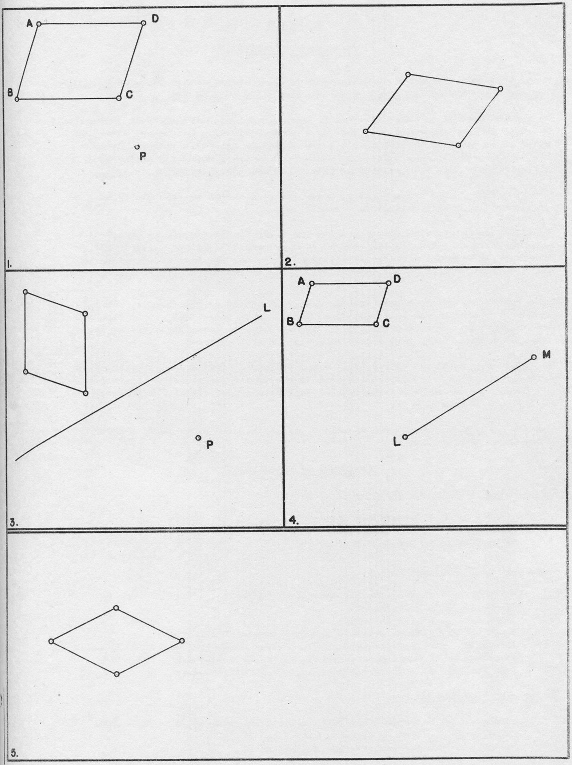
FIG. 1. Draw through P the line parallel to the diagonal BD of the given parallelogram. (Hint: B, D, and the center of the parallelogram offer a bisected segment parallel to the desired line.)

FIG. 2. Construct a parallelographic network of lines. (Hint: Bisect the sides.)

FIG. 3. Construct the parallel line to L through P. (Hint: See Plate 45,6.)

FIG. 4. Divide the given segment LM into three equal parts. (Hint: Extend the segment AD to three times its length by Fig. 2 above. Project these equal segments onto a line parallel to LM and complete the construction with a second projection.)

FIG. 5. The given parallelogram here is a rhombus. Construct a rectangular network of lines. Beyond drawing these perpendiculars in this fixed direction, does the rhombus offer any possibilities in addition to those obtained with the general parallelogram?


A
D
B
C
P
1.
2.
L
P
3.
A
D
B
C
M
L
4.
5.

## SECTION VIII

## THE ASSISTED STRAIGHTEDGE

The constructions of this section are those that may be accomplished by the Straightedge and Collapsible Compasses, the Straightedge and Rigid Compasses, the Straightedge and Rigid Dividers.

The compasses of Euclid and Plato differs from the modern instrument in that theoretically it collapses when lifted from the plane. Thus it may not be used to transfer distances from one part of the plane to another and may be used to establish a circle only when given its center and a point upon its circumference. The collapsible compasses is proved equivalent to the modern compasses by showing that it is possible to draw a circle whose radius is not given in position. (See Plate 48,5.)

(After completing the work of Plate 48, refer to Plates 2 and 3. See how you would need to change your construction there if required to use the collapsible compasses.)

The Rigid Compasses has a fixed opening and may be used to draw circles with arbitrary centers all having the same radii. Obviously, this would put us in possession of a fixed circle, which, with the Straightedge, is equivalent to Straightedge and Variable Compasses. (See Section VII.) However, the constructions will be found somewhat different from those of Poncelet-Steiner and offer added interest.

Pappus reports that the ancient Greeks were themselves concerned with the Rigid Compasses; Mascheroni found it of practical use when he employed several compasses in his constructions, laying a fixed one aside until he had to use the same radius again. This was claimed to produce greater accuracy than setting and resetting a single pair of compasses for circles of different radii.

The Rigid Dividers has a fixed opening and may be used to transfer a constant length from one portion of the plane to another. If the carrying operation be restricted to placing the fixed length upon a line already drawn, the system of straightedge and rigid dividers is not equivalent to straightedge and compasses. (Compare Plate 59.)

Since the unit of measure is arbitrary, we select as the unit the length of the opening of either the Rigid Compasses or Rigid Dividers.

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## THE STRAIGHTEDGE AND COLLAPSIBLE COMPASSES

FIG. 1. Draw the perpendicular from P to the line L.

FIG. 2. Draw the parallel from P to the line L. (Hint:* With center at a selected point A on L draw the circle A(P) which meets L in B. Draw circle B(A) meeting P(A) in X.) Compare other known methods of drawing parallels. How many circles are used in the construction?________________. How many radii?________________.

FIG. 3. Bisect the given angle.

FIG. 4. Transfer the distance AB to the line L. (Hint: Locate the intersection of L and AB.)

FIG. 5. The collapsible compasses is equivalent to the modern compasses if it is possible to draw the circle with center at O and radius AB, where neither A nor B coincides with O. Make the construction. (Hint: Draw circles O(A) and A(O) meeting at C. Draw lines OC and AC. Draw circle A(B) meeting AC in D. Draw circle C(D) meeting OC in D'.)

---

* A(P) indicates the circle with center A and passing through P.

Plate 48.

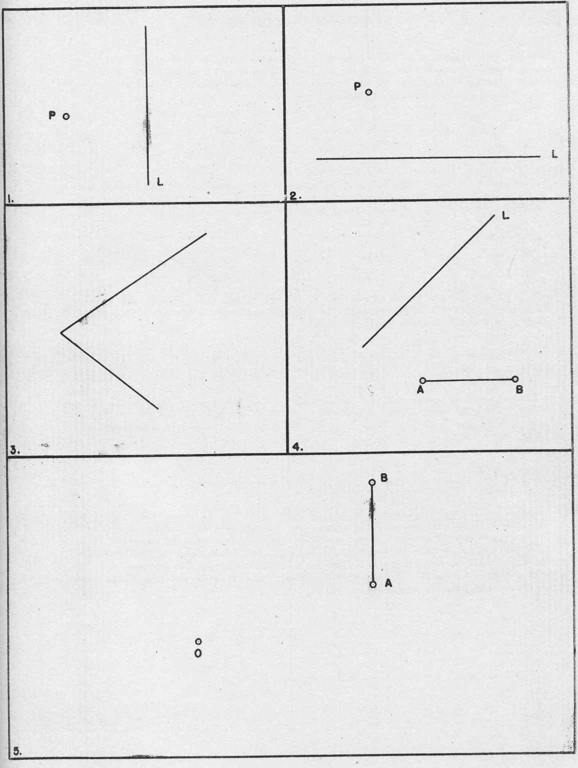


FIG. 1. Draw the parallel to L through P. (Hint: Assuming the opening of the compasses to be greater than the distance from P to L, draw the circle with center at P meeting L in X. With X as center, draw the circle cutting the line PX in O. With O as center, cut L in Y. With Y as center, cut OY in Q. Then PQ is the desired parallel. Why?)

FIG. 2. Erect the perpendicular to L at P. (Hint: With P as center, draw the circle cutting L in A, B. With B as center, draw the semicircle meeting the first circle in C. With C as center, draw the circle meeting the line BC in X. Then XP is the desired perpendicular. Why?)

FIG. 3. Divide the given segment AB into three equal parts. (Hint: Erect perpendiculars to AB at A and B. Upon each of these perpendiculars, lay off three equal segments in opposite directions. Their joins will meet AB in the specified points.)

FIG. 4. Extend the segment AB to C such that AB = BC. (Hint: Erect the perpendicular to AB at A upon which two equal segments AX, XY are laid off. The parallel to XB through Y meets AB extended in C.)

FIG. 5. Find the intersections of the given line L and the hypocircle O(A). (Hint: In order to aid the student, we have already drawn the circle with the rigid compasses having its center at O. Proceed as follows. Let B be the foot of the perpendicular from O to L. Draw the lines AB and AO, the latter meeting the drawn circle in C. Draw the parallel to AB through C meeting OB in D. Through D draw the parallel to L, which meets the drawn circle in X and Y. The lines OX and OY meet L in the desired points of intersection. Why?)

FIG. 6. From P upon L lay off a length equal to the given segment AB.

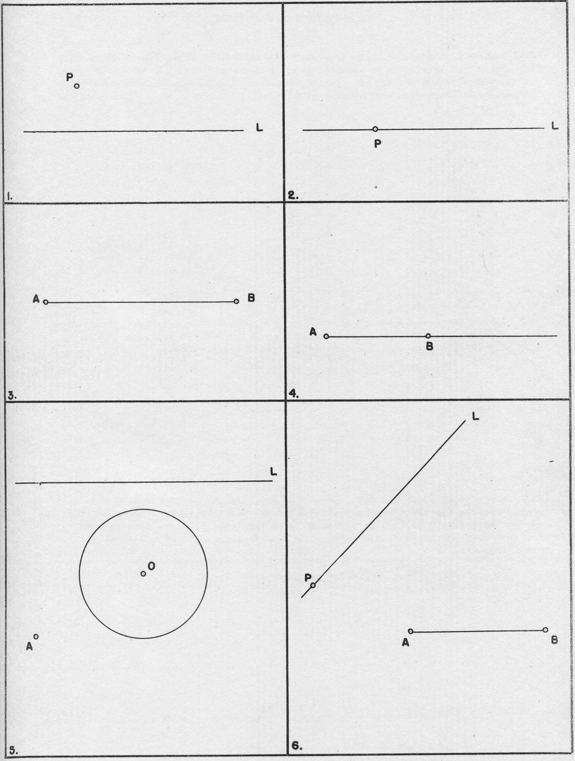

P
L
1.
L
P
2.
A
B
3.
A
B
4.
L
O
A
5.
L
P
A
B
6.

FIG. 1. Find the intersections of the two hypocircles, O(A) and O'(A'). (Hint: Proceed to find the radical axis by first transferring the distance O'A' to the line through O' parallel to OA. Then locate the external centers of similitude. For convenience and uniformity, the circle drawn with the rigid compasses is already given with its center at O.)

FIG. 2. Draw the line through P parallel to L. (Hint: A solution differing from that of the previous plate is as follows: with the rigid compasses, draw the circle with center at P cutting L in A and B. Produce BP to meet the circle in C. Then construct the bisector of angle CPA.)

FIG. 3. Draw the perpendicular to L at P. (Hint: A solution differing from that of the previous plate is as follows: draw the circle with center at P cutting L in A and B. Draw the semicircle with A as center meeting the first circle in C. Draw the circle with center at C. This meets the first circle in D. Draw the circle with center at D. These last two circles meet in X such that PX is the required perpendicular.)

Plate 50

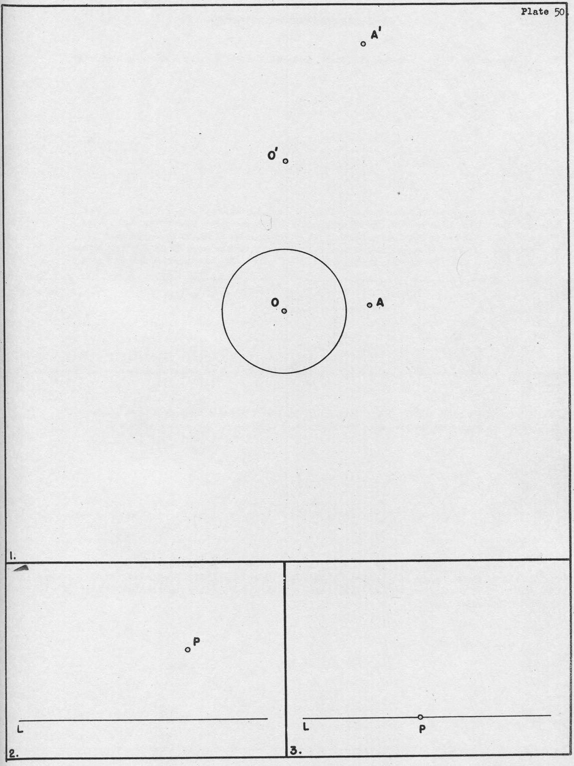
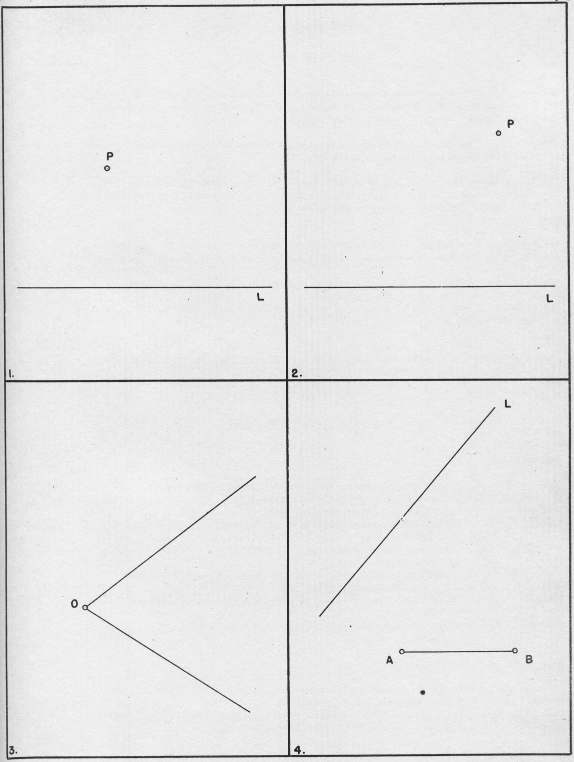


FIG. 1. Draw the parallel to L through P. (Hint: Apply the dividers twice to L obtaining a bisected segment.)

FIG. 2. Draw the perpendicular from P to L. (Hint: Lay off the bisected segment AOB upon L. Then with one point of the dividers at O, lay off arbitrarily two further points C, D. Thus, A, B, C, D lie on a circle with center at O, and accordingly AC is perpendicular to BC; BD is perpendicular to AD. Therefore, if AD and BC be produced to meet in E then the intersection of BD and AC is the orthocenter of triangle ABE. The altitude through E is perpendicular to AB. Its parallel through P is the desired line.)

FIG. 3. Bisect the given angle. (Hint: Starting at the vertex O, lay off two consecutive lengths OAB, OCD on the sides with the dividers. The intersection of AD and BC lies on the bisector.)

FIG. 4. Transfer the given segment AB onto the given line L. (Hint: Draw the line AB' parallel to L. Lay off the divider length AX upon AB and AY upon AB'. Then triangle AXY is isosceles. Draw the parallel to XY through B meeting AB' in B'. Then AB = AB'. Two parallel lines through A and B' meet L in the desired length.)


P
L
1.
P
L
2.
O
3.
L
A
B
4.

## SECTION IX

### PARALLEL AND ANGLE RULERS

The Parallel Ruler is defined as an instrument of indefinite length having two parallel straight edges. The width of the Ruler shall be designated as the unit of measurement. It shall be used in the following two ways:

I. To determine the line through two given points and its parallel at a unit's distance (i.e., the line determined by the other edge of the ruler).

II. To determine a line through each of two given points, A, B, at a unit's distance apart. (The ruler may be placed so that an edge passes through each of the two points, A, B if the distance AB is greater than unity. This may be done in two ways.)

The Angle Ruler is defined as an instrument of indefinite extent having two straight edges which form a constant angle. Besides its service as a simple straightedge, it shall be used:

I. To determine a line making the fixed angle with a given line.

II. To determine lines through two given points making with each other the fixed angle (i.e., with an edge through each point).

IT IS SHOWN HEREIN THAT THESE TWO TOOLS ARE EQUIVALENT TO THE STRAIGHTEDGE AND COMPASSES AND ARE EACH CAPABLE OF MAKING ALL CONSTRUCTIONS OF PLANE EUCLIDEAN GEOMETRY.

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| | | |
|---|---|---|
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Angle Ruler:

| | | |
|---|---|---|
| Adler, A. | : | Theorie der geometrischen Konstruktionen, Leipzig, (1906) 132-138; 177-180. |
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THE PARALLEL RULER

FIG. 1. Draw the perpendicular to L at P. (Hint: Place the ruler in an arbitrary position with one edge passing through P. Draw along the edges. Then move it parallel to itself so that the other edge passes through P. This gives a bisected segment XPY. Now place the ruler so that a different edge passes through the points X and Y. This may be done in two ways - the two positions determining a rhombus with one diagonal as the line L, the other diagonal passing through P.)

FIG. 2. Construct a rectangular network of lines.

FIG. 3. Draw the parallel to L through P. (Hint: This may be done in a number of ways, two of which are as follows. Either obtain a bisected segment upon L and follow Plate 29,6; or place one edge of the ruler along L and draw along the other edge thus obtaining two parallel lines to which the construction of Plate 31,6 may be applied.)

FIG. 4. A very simple construction for the parallel to L through P is shown. Place the ruler with one edge through P and move so as to establish the equidistant points A, B, C, D, E, upon L. Draw AP, meeting the middle line in F. Draw FE meeting the line through D in Q. Then PQ is the required parallel. Why?)

FIG. 5. Draw the perpendicular to L through P. (Hint: First draw a line through P parallel to the given line, then apply Fig. 1.)

FIG. 6. Determine other points upon the hypocircle with center O and passing through A. (Hint: As in Fig. 1, locate the other extremity B of the diameter AOB. Then find the intersections of perpendiculars dropped from A upon arbitrary lines through B.)

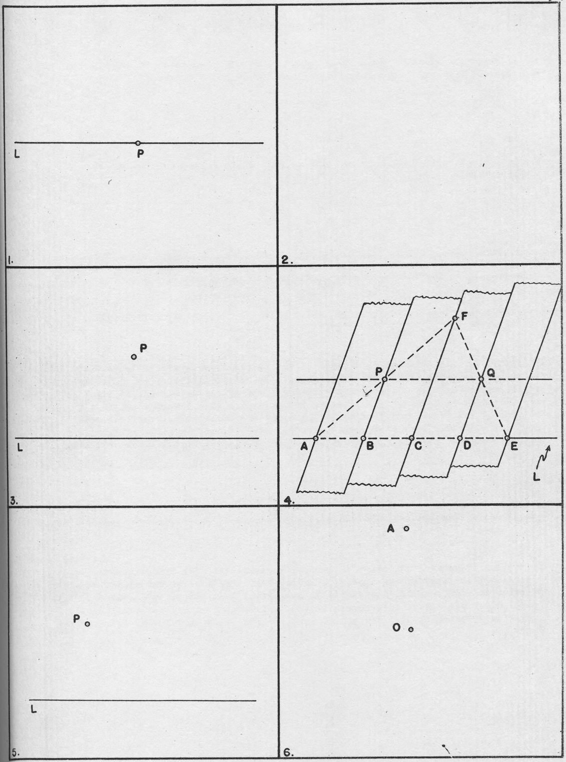

L
P
1.
2.
P
L
3.
F
P
Q
A
B
C
D
E
L
4.
P
L
5.
A
O
6.

FIG. 1. Bisect the angle AOB. (Hint: Place the ruler first with one edge along OA, then with one edge along OB. This establishes a rhombus whose diagonal is the desired bisector.)

FIG. 2. Locate the point D on OC such that OA/OB = OC/OD. (Hint: Join A and C; then draw its parallel from B.)

FIG. 3. Transfer the distance OA onto OL. (Hint: Construct as in Fig. 1 the rhombus upon the sides of the given angle. Then draw through A the parallel to a diagonal of the rhombus.)

FIG. 4. Let the width of the ruler be the unit. Find the points of intersection of the line L with the hypo <u>unit</u> circle with center at O. (Hint: Employing the idea of poles and polars, select any point P upon L. From P draw the tangents to the unit circle by placing the ruler with one edge through P and the other edge through O. Draw the perpendiculars to these tangents from O and call the points of tangency thus found A and B. Let Q be the point of intersection of the perpendicular from O upon L with AB. Then Q is the polar of L with respect to the unit circle. Accordingly, place the ruler between Q and O and establish the tangents from Q which meet L in the desired points. Explain further.)

FIG. 5. **Find** the points of intersection of L with the hypocircle O(A). (The distance OA is not equal to the width of the ruler which is assumed to be unity.) (Hint: Find B the point of intersection of the unit circle with OA. Let C be the foot of the perpendicular from O upon L. Draw AC and its parallel from B which meets OC in D. Draw the parallel to L through D and find its intersections, by Fig. 4, with the unit circle. If these be $P_1$ and $P_2$, then $OP_1$ and $OP_2$ meet L in the desired points. Explain.)

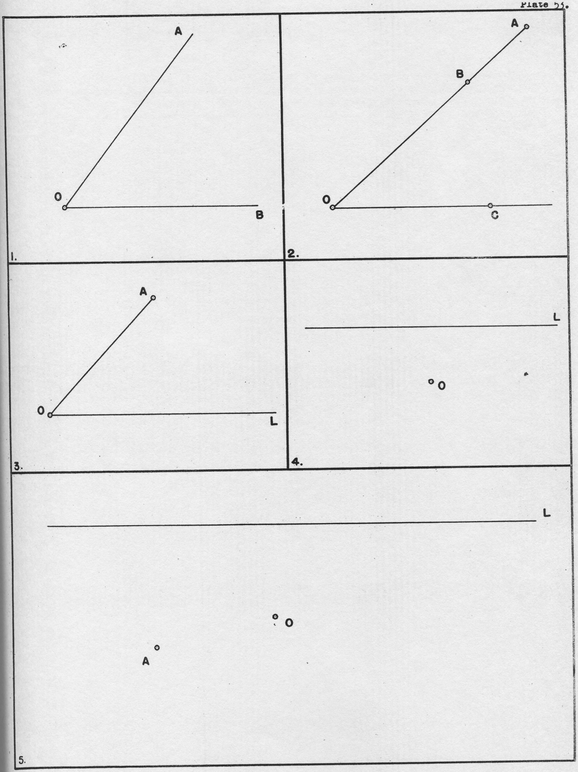

A
O
B
1.
A
B
O
C
2.
A
O
L
3.
L
O
4.
L
O
A
5.

FIG. Find the intersections of the two hypocircles O(A) and O'(A'). (Hint: Proceed to establish their radical axis, then find the intersection of this axis with either circle according to the previous plate. First transfer the distance O'A' onto a line through O' parallel to OA. Then locate the external center of similitude of the given circles. Complete the construction and explain.)

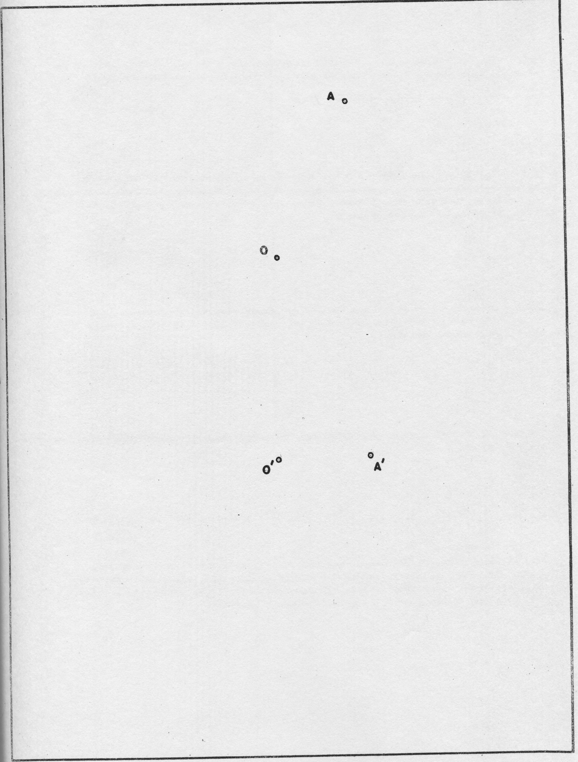

A
O
O'
A'

FIG. 1. Draw the parallel to L through P. (Hint: With one edge of the ruler along L, draw along the other edge passing through P. Then slide the ruler along this line until the first edge passes through P.)

FIG. 2. Draw the perpendicular from P to L. (Hint: Select two arbitrary points A and B upon L. Place the ruler in two opposite positions on one side of L so that the vertex is at A and B. Then reflect the positions in L. This produces a rhombus one of whose diagonals is perpendicular to L. A parallel through P is the desired line.)

FIG. 3. Extend the segment AB to C such that AB = BC. (Hint: Place the ruler in an arbitrary position with its edges passing through A and B. Now place it parallel to its original position with the other edge passing through B.)

FIG. 4. What is the path of the vertex of the ruler if its edges remain in contact with the fixed points A and B? Explain.

FIG. 5. Locate arbitrary points upon the hypocircle O(A). (Hint: Locate B, the other extremity of the diameter AOB. Place the ruler with vertex at B and one edge along AOB. The line determined by the other edge meets its perpendicular from A in P, a point of the hypocircle. If the ruler is now moved with these edges touching A and P, the vertex describes the hypocircle.)

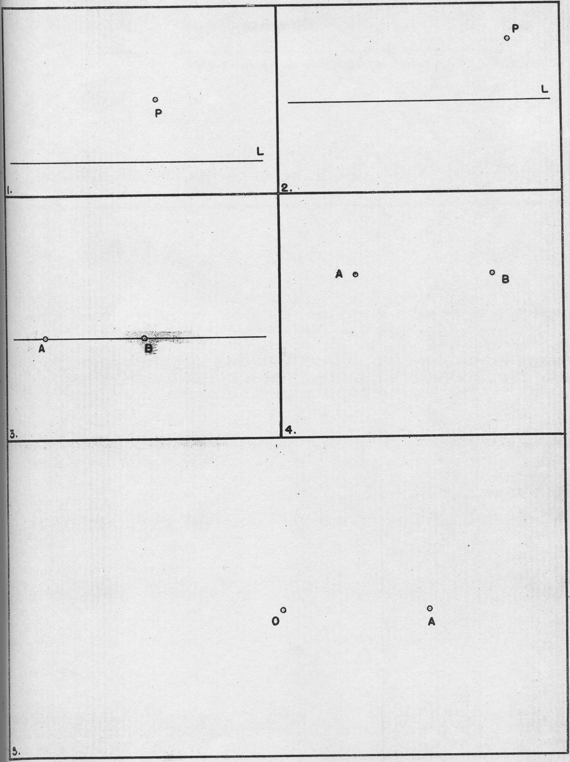
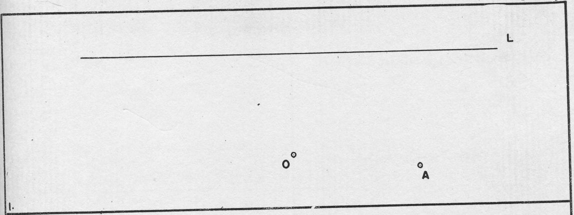

P
L
1.
P
L
2.
A
B
3.
A
B
4.
O
A
5.

FIG. 1. Find the intersections of the line L with the hypocircle O(A).

FIG. 2. Find the intersections of the two hypocircles O(A) and O'(A'). (Hint: Establish their radical axis.)


L
O
A
1.

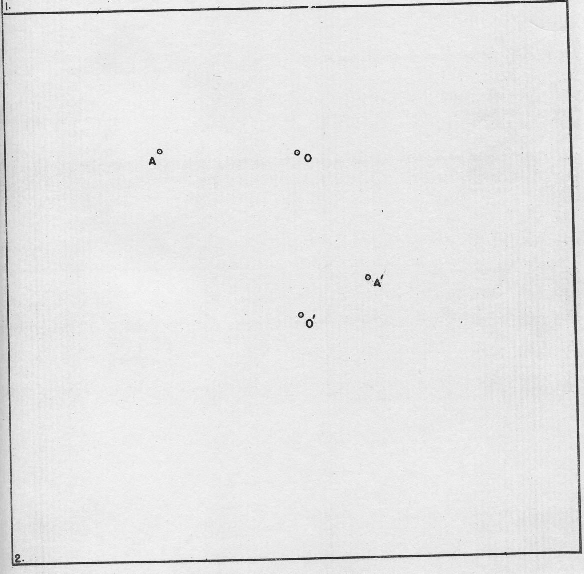

A
O
A'
O'
2.

FIG. 1. Establish the perpendicular from P to L.

FIG. 2. Draw the parallel to L through P.

FIG. 3. Extend the segment AB to C such that AB = BC.

FIG. 4. The ruler moves with its two edges in contact with the fixed points A and B. Determine the path of the vertex.

FIG. 5. Determine the intersections of the line L with the hypocircle O(A).

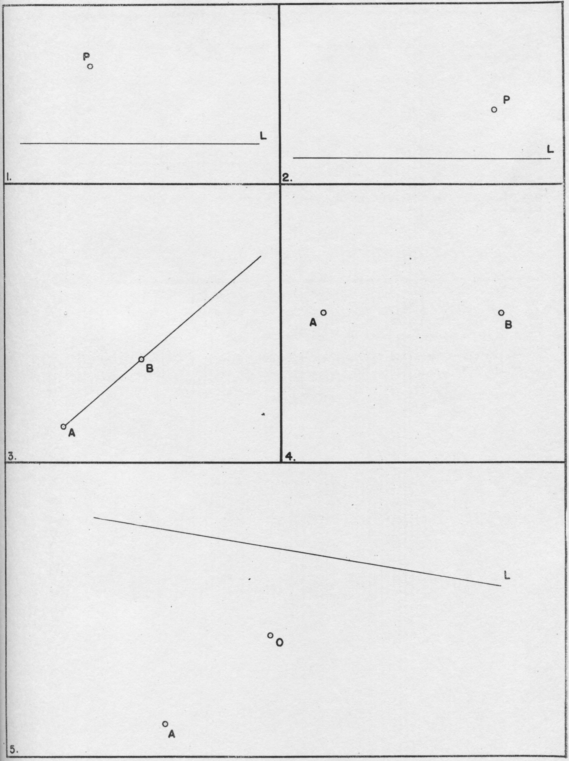
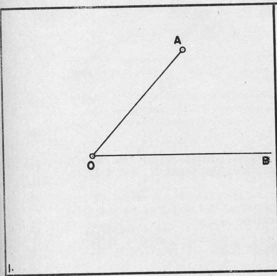
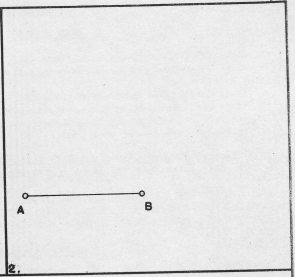

P
L
1.
P
L
2.
B
A
3.
A
B
4.
L
O
A
5.

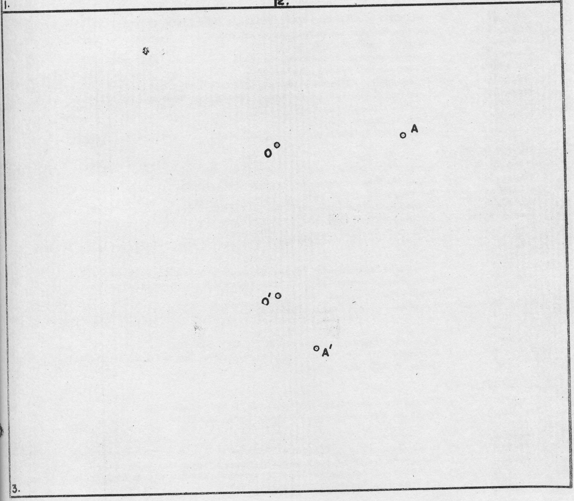
FIG. 1. Transfer the distance OA onto OB and bisect the angle AOB. (Hint: Locate the other extremity C of diameter AOC.)

FIG. 2. Construct an equilateral triangle with side AB. (Hint: Locate the point C on AB extended such that AB = BC. Place the ruler with its edges through A and B so that its vertex lies upon the perpendicular bisector of AB.)

FIG. 3. Find the intersections of the hypocircles O(A) AND O'(A').


A
O
B
1.


A
B
2.


A
O
O'
A'
3.

## SECTION X

### HIGHER TOOLS AND QUARTIC SYSTEMS

The Marked Ruler is a straightedge of indefinite length upon the edge of which two arbitrary points, P, Q, are marked. We shall take the distance PQ as the unit of measure. The ruler shall be used in the following three ways:

1. To establish the line upon two given points and to mark upon this line successive unit lengths;
II. To fix Q at a given point of the plane and rotate the ruler until P falls upon a given line;
III. With the straightedge passing through a given point of the plane, to move Q along a given line until P falls upon a second given line.

IT IS SHOWN THAT THE MARKED RULER IS EQUIVALENT TO STRAIGHTEDGE AND COMPASSES IF USED UNDER ASSUMPTIONS I AND II; AND IF USED UNDER I, II, AND III IT IS CAPABLE OF SOLVING ALL PROBLEMS OF A QUARTIC NATURE.

IT IS ALSO SHOWN THAT THE CARPENTER'S SQUARE, THE TOMAHAWK, TWO RIGHT ANGLE RULERS, AND THE COMBINATION OF COMPASSES WITH IMMOVABLE CONIC ARE EACH QUARTIC TOOLS OR SYSTEMS. THE IMMOVABLE CONIC WITH STRAIGHTEDGE COMPRISES A QUADRATIC SYSTEM (EQUIVALENT TO STRAIGHTEDGE AND COMPASSES).

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We first consider the possibilities of the Marked Ruler employed under Assumption 1. The distance between the points P and Q upon the Ruler is taken as the unit distance.

FIG. 1. Draw the parallel to L through P.

FIG. 2. Draw the perpendicular to L through P. (Hint: See Plate 51,2.)

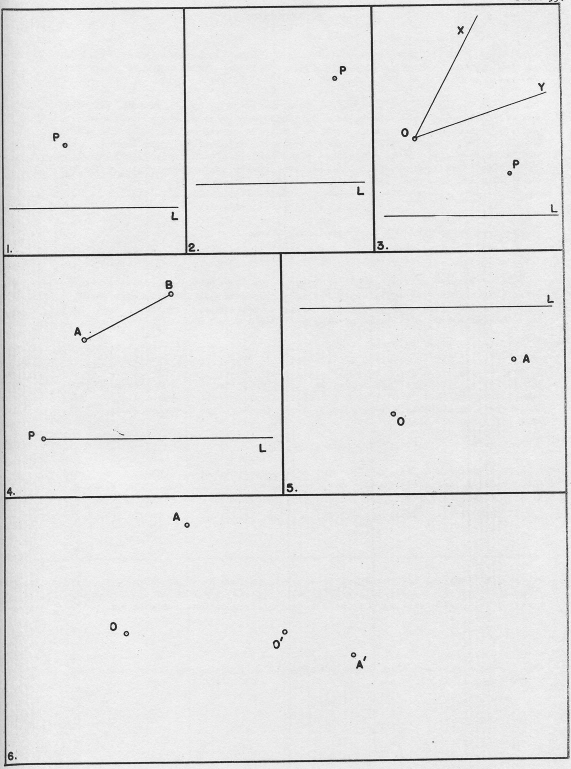
FIG. 3. Through P draw the line making the given angle XOY with the line L. (Hint: From any selected point B on side OY drop perpendiculars upon OX and upon the line through O parallel to L meeting them in A and C respectively. Draw the perpendicular from O to AC. This is parallel to the desired line through P. Make the construction and explain.)

FIG. 4. Transfer the distance AB ($\neq 1$) to the line L from the given point P. (Hint: See Plate 51,4.)

The Marked Ruler under Assumption I is capable of constructing parallels and perpendiculars, transferring distances, bisecting and multiplying angles. Its powers are not as extensive as those of the compasses alone. The possibilities are considerably amplified, however, if we employ Assumption II, as follows:

FIG. 5. Find the intersections of the line L with the hypocircle O(A) where $OA \neq 1$.

FIG. 6. Find the intersections of the two hypocircles O(A) and O'(A'). (Hint: Transfer the distance O'A' onto a line through O' parallel to OA, locate the external center of similitude, and proceed to establish the radical axis as in previous plates. See Plate 50,1.)


P
L
1.
P
L
2.
X
Y
O
P
L
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B
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P
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4.
L
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O
5.
A
O
O'
A'
6.

Under Assumption III, the Marked Ruler moves with one of its points upon a given line or circle while the edge, PQ extended, passes through some fixed point. We inquire into the analytical implication of this assumption.

FIG. 1. Find the path of P as Q moves along the given line L with the edge, PQ extended, passing through the fixed point O at a distance $\underline{a}$ from L. (Hint: Since the points of the ruler are arbitrarily named, the problem implies that P might move along L and it is required also that we find the path of Q. Accordingly, the problem is equivalent to the following. The unit circle moves with its center Q on the line L. Find the path of the intersections of this circle with the line joining its center and a fixed point O. Take the fixed point O as the pole of a system of polar coordinates and the line through O parallel to L as polar axis. Then P has the coordinates $(r, \theta)$ and directly:

$$r = 1 + OQ = 1 + a.\csc \theta.$$

If the point R be considered with coordinates $(r, \theta)$, we have

$$r = -1 + a.\csc \theta.$$

These are the two branches of the Conchoid of Nicomedes. The Marked Ruler arrangement here could be replaced by a circular wheel rolling upon a line one unit below the given line L, to the center of which is attached a straightedge passing always through a sleeve pivoted at the fixed point O. Obtain the rectangular equation of this curve by taking X- and Y-axes through the point O.

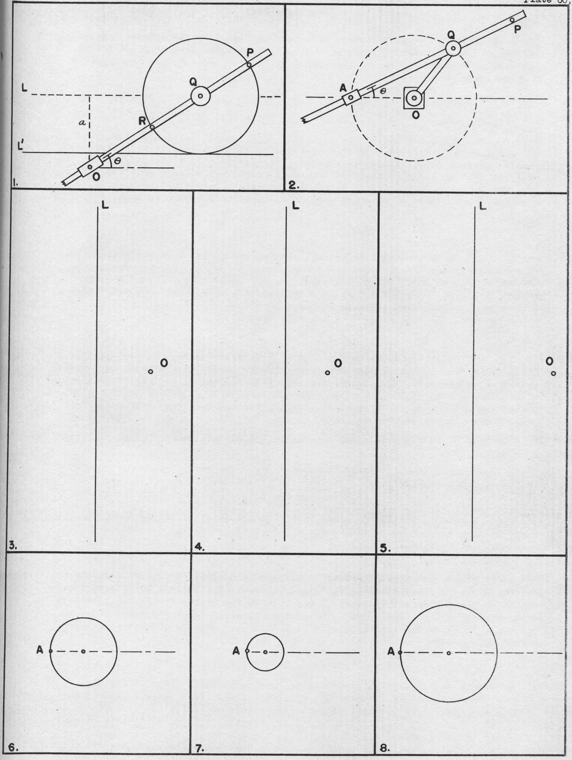
FIG. 2. Find the path of P as Q moves along the given circle of diameter $\underline{a}$, with the edge, PQ extended passing through the fixed point A lying on the circle. (Hint: Take the fixed point A as pole, the line through A and the center O of the circle as polar axis. Let P have the coordinates $(r, \theta)$. Then, since the distance $AQ = a.\cos \theta$,

$$r = 1 + a.\cos \theta$$

This will be recognized as the equation of the Limacon of Pascal, introduced in Plate 17. The Marked Ruler arrangement here could be replaced by the system of two linked bars shown. The bar PQ slides through a sleeve pivoted at A. Obtain the rectangular equation of the curve by choosing X- and Y-axes through the point A.

FIG. 3, 4, 5. Sketch the Conchoids defined by the fixed lines L and the points O. In Fig. 3, take $\underline{a} = 1$; in Fig. 4, $0 < \underline{a} < 1$; in Fig. 5, $\underline{a} > 1$. In sketching, draw a series of unit circles with their centers on L and mark their points of intersection with lines joining their centers and the point O.

FIG. 6, 7, 8. Sketch the Limacons defined by the fixed points A and the given circles of diameter $\underline{a}$, passing through A. In Fig. 6, take $\underline{a} = 1$; in Fig. 7, $\underline{a} = 1/2$; in Fig. 8, $\underline{a} = 3/2$.


L
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Q
a
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A
7.
A
8.

FIG. 1. Let us assume the ability to move P along one given line and Q at the same time along another such that the line PQ, extended if necessary, shall pass through a given fixed point. We take the given lines as coordinate axes and the fixed point as (h, k). Then if the distance PQ is 1 and its intercepts are x, y:

$$x^2 + y^2 = 1.$$

From similar triangles, $y = kx/(x - h)$. Using this to eliminate y, we have:

$$x^4 - 2hx^3 + (h^2 + k^2 - 1)x^2 - 2hx - h^2 = 0.$$

Thus, since there may be four real solutions here, there are four possible positions of the segment PQ. Draw them. Can you locate the point (h, k) such that there will be but two real solutions? No real solutions? This fitting of the segment PQ between two given curves is known as the insertion principle.

FIG. 2. The segment PQ is here inserted between a given line and a circle so that it passes through the fixed point (h, k). This also leads to a quartic equation (as may be verified by an appropriate selection of reference axes). Sketch in the other three positions of the segment PQ. Locate the point (h, k) such that there will be but two real solutions of the quartic and thus but two positions of the segment; such that there will be no solution.

FIG. 3. The ancient and famous problem of Trisecting the Angle has for its algebraic interpretation an equation of the third degree. For, let the given angle be AOB = $3\theta$ whose cosine is a. Suppose that one of the trisecting lines, OT, is already drawn. Let OA = 1 and draw AC parallel to OT. Then $\angle$ ACO = $\theta$. Locate the point D so that OD = 1. Then since triangle AOD is isosceles, $\angle$ DAO = $\angle$ ADO = $2\theta$. But angle ADO is the exterior angle of triangle ODC and, since $\angle$ DCO = $\theta$, $\angle$ DOC = $\theta$. Thus DC = 1.

Let DA = 2y, OC = x. From similar triangles CMD, CNA, and CLO, all right triangles with equal angles at C, we have:

$$x/2 = (x + a)/(1 + 2y) = (1 + y)/x,$$

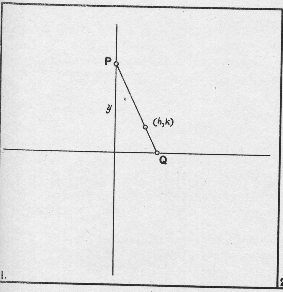
which gives on eliminating y:

$$x^3 - 3x - 2a = 0 \qquad \text{where } |a| \leqq 1.$$

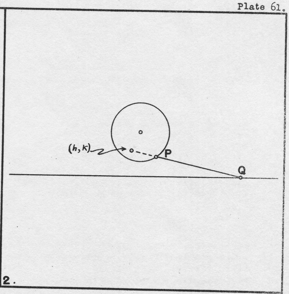
Accordingly, the problem of trisection is equivalent to the solution of an algebraic equation of third degree, for if we can construct the value x that satisfies this equation the angle is trisected geometrically by drawing parallel lines.

FIG. 4. Trisect the given angle AOB by means of the insertion principle. (Hint: Let the distance between the marks P, Q on the ruler be OL, chosen arbitrarily on OB. Bisect OL to obtain the point K. Draw KM parallel to OA and KN perpendicular to OA. Place the segment PQ so that P falls on KM with PQ extended passing through O. Now move the ruler until Q falls on KN. When this happens, the line PQ trisects $\angle$ AOB. Draw the figure and prove by inspecting angles. That is, if H be the midpoint of PQ, then HK = OK = HQ = HP.

FIG. 5. Trisect the given angle AOB by means of the insertion principle. (Hint: Take OB as the arbitrary distance PQ and draw the circle O(B), which meets OA in D. Place the ruler so that Q falls on BO extended, P upon the circle, with PQ extended passing through D.)



1.



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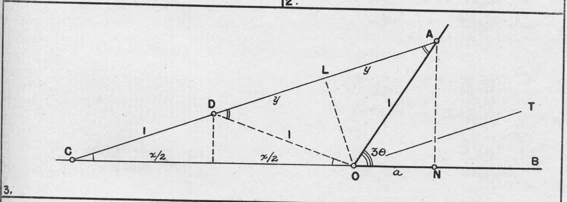

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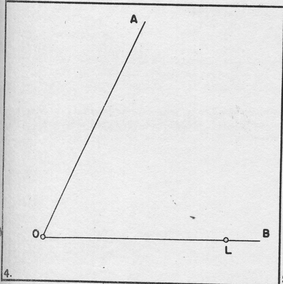

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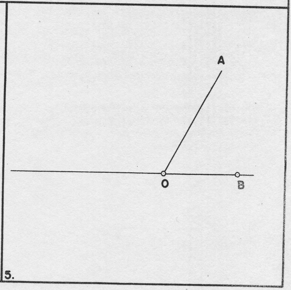

5.

## QUARTICS

Consider $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are given or constructed geometric lengths. If we let $x = y - a/4$, this equation reduces to

$$y^4 + Ay^2 + By + C = 0 \qquad \ldots\ldots\ldots\ldots(2)$$

Calculate the following in terms of a, b, c, d:

A =
B =
C =

All of these will be found as quantities constructible from a, b, c, d by straightedge and compasses.

Now let
$$y = u + v + w$$
so that
$$y^2 = u^2 + v^2 + w^2 + 2(vw + uw + uv)$$
$$y^4 = (u^2 + v^2 + w^2)^2 + 4(u^2 + v^2 + w^2)(vw + uw + uv) + 4(v^2w^2 + u^2w^2 + u^2v^2) + 8uvw(u + v + w).$$

These quantities substituted in (2) give:

$$(u^2 + v^2 + w^2)^2 + 4(v^2w^2 + u^2w^2 + u^2v^2) + A(u^2 + v^2 + w^2) + C +$$
$$2(vw + uw + uv)[2(u^2 + v^2 + w^2) + A] + (u + v + w)(8uvw + B) = 0 \qquad \ldots\ldots\ldots\ldots(3)$$

By introducing these three quantities u, v, w, in place of y we have allowed ourselves considerable freedom of selection. We exercise this freedom in choosing:

$$uvw = -B/8$$
$$u^2 + v^2 + w^2 = -A/2$$
$$v^2w^2 + u^2w^2 + u^2v^2 = (A^2 - 4C)/16 \qquad \ldots\ldots\ldots\ldots(4)$$

So that equation (3) will be satisfied. Now we may think of the quantities $u^2$, $v^2$, $w^2$ as the roots of a cubic:

$$(z - u^2)(z - v^2)(z - w^2) = 0,$$

or

$$z^3 - (u^2 + v^2 + w^2)z^2 + (v^2w^2 + u^2w^2 + u^2v^2)z - u^2v^2w^2 = 0.$$

In the light of equations (4), this cubic may be written as:

$$z^3 + Az^2/2 + (A^2 - 4C)z/16 - B^2/64 = 0. \qquad \ldots\ldots\ldots\ldots(5)$$

Thus far we have reduced the original quartic to an equivalent equation of third degree. This equation is called the resolvent cubic. (Compare: The solution of a quadratic depends on a resolvent linear equation; the solution of a cubic depends on a resolvent quadratic.) If the three roots of (5) are $z_1$, $z_2$, $z_3$, then

$$u^2 = z_1 \qquad \text{or} \qquad u = \pm\sqrt{z_1}$$
$$v^2 = z_2 \qquad\qquad v = \pm\sqrt{z_2}$$
$$w^2 = z_3 \qquad\qquad w = \pm\sqrt{z_3}$$

But not all combinations of the algebraic signs here are permissible. These values of u, v, w must satisfy the set of equations (4).

Thus

$$x_1 = \sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3} - a/4 .$$

$$x_2 = \sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3} - a/4.$$

$$x_3 = -\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} - a/4.$$

$$x_4 = -\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3} - a/4.$$

Now, since equation (5) has coefficients which are constructible from a, b, c, d by straightedge and compasses, this reduction of a quartic to its resolvent cubic demands no other tools. Write equation (5) for the sake of brevity as:

$$z^3 + Dz^2 + Ez + F = 0,$$

(where D, E, F are straightedge and compasses constructible) and let $z = s - D/3$.

The cubic becomes
$$s^3 + Hs + K = 0. \qquad \ldots\ldots\ldots\ldots(6)$$

where

$$H = E - D^2/3$$
$$K = 2D^3/27 - DE/3 + F ,$$

quantities which are themselves constructible in the same sense. Now let $s = Kt/H$. Equation (6) becomes:

$$t^3 + m(t + 1) = 0 \qquad \ldots\ldots\ldots\ldots(7)$$

where $m = H^3/K^2$. This quantity m is a constructible function of the constructible quantities H and K.

Accordingly,

THE GENERAL QUARTIC IS ALWAYS REDUCIBLE TO A RESOLVENT CUBIC DEPENDENT UPON A SINGLE CONSTANT, WHEREIN THE ONLY ALGEBRAIC OPERATIONS INVOLVED ON THE GIVEN COEFFICIENTS OF THE QUARTIC ARE THOSE THAT ARE EQUIVALENT TO STRAIGHTEDGE AND COMPASSES CONSTRUCTIONS.

Find the quantity <u>m</u> in terms of the given quantities <u>a</u>, <u>b</u>, <u>c</u>, <u>d</u>.

## CUBICS

Consider the cubic (Equation 7 of Page 153):

$$t^3 + mt + m = 0. \qquad \text{............(1)}$$

where m is a given or constructed length and, of course, real. Let $t = u + v$. We have:

$$(u^3 + v^3) + m + (3uv + m)(u + v) = 0,$$

an equation that is satisfied if

$$\left.\begin{array}{r} u^3 + v^3 = -m \\ uv = -m/3. \end{array}\right\} \qquad \text{............(2)}$$

If v be eliminated between these last two equations, we have:

$$27u^6 + 27mu^3 - m^3 = 0.$$

This equation is a <u>resolvent</u> quadratic in the quantity $u^3$. A solution is

$$u^3 = (m/2)[-1 + \sqrt{(1 + 4m/27)}] = R \qquad \text{............(3)}$$

Show that the other root of the quadratic is $v^3 = -m^3/27u^3$ [from (2)].

The three cube roots of (3) are $u$, $\omega u$, $\omega^2 u$ where $\omega^3 = 1$ and the corresponding values of v (such that $uv = -m/3$) are: $-m/3u$, $-m/3\omega u$, $-m/3\omega^2 u$.
Thus, since $t = u + v$, the three roots of (1) are:

$$t_1 = u - m/3u; \qquad t_2 = \omega u - m/3\omega u; \qquad t_3 = \omega^2 u - m/3\omega^2 u. \qquad \text{............(4)}$$

The Discriminant, $\Delta$, of an algebraic equation is defined as the square of the product of the differences of its roots taken in pairs. For the cubic (1) above:

$$\Delta = [(t_2 - t_3)(t_3 - t_1)(t_2 - t_1)]^2 \qquad \text{............(5)}$$

which from (4) is:

$$\Delta = (\omega u - \omega^2 u - m/3\omega u + m/3\omega^2 u)^2(u - \omega^2 u - m/3u + m/3\omega^2 u)^2(u - \omega u - m/3u + m/3\omega u)^2$$

$$= (\omega - \omega^2)^2(u + m/3u)^2(1 - \omega^2)^2(u + m/3\omega^2 u)^2(1 - \omega)^2(u + m/3\omega u)^2,$$

and since $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$,

$$\Delta = -27(u + m/3u)^2(u + m/3\omega^2 u)^2(u + m/3\omega u)^2 = -27(u^3 + m^3/27u^3)^2 = -27(u^3 - v^3)^2.$$

From equation (3) this is:

$$\Delta = -m^2(27 + 4m) \qquad \text{............(6)}$$

The value of this discriminant enables us to tell the character of the roots in advance of the solution. For, from an inspection of (5):

I. All roots are real and unequal if $\Delta > 0$, i.e. $27 + 4m < 0$ or $m < -27/4$.

II. If two or more roots are equal, $\Delta = 0$, and either $m = 0$ or $m = -27/4$.

III. If but one root is real, the other two are conjugate complex and their difference is pure imaginary. Thus $\Delta$ is negative and $27 + 4m > 0$, $m > -27/4$.

Returning to $t^3 + mt + m = 0$, let $t = r.\sqrt{(-m/3)}$.
which is a real transformation only if $m < 0$. Substitution gives:

$$r^3 - 3r - 3\sqrt{(-3/m)} = 0.$$

This equation will be a <u>Trisection Equation</u> (see Plate 61,3) if the constant term lies between -2 and +2. That is, if

$$-2 \leq -3\sqrt{(-3/m)} \leq +2 .$$

The values of $m$ that satisfy this inequality are:

$$m \leq -27/4.$$

But this is just the condition that the original cubic have <u>all real roots</u>. Accordingly,

EVERY CUBIC EQUATION WHICH HAS THREE REAL ROOTS CAN BE SOLVED BY A MARKED RULER IN THE TRISECTION MANNER.

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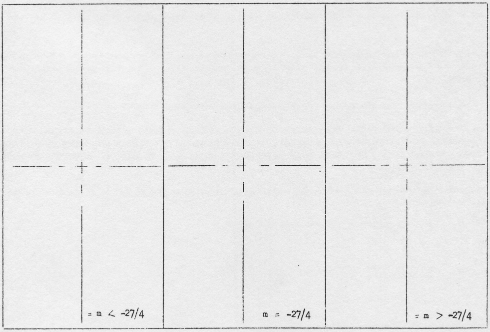
In the three spaces below, sketch the function $t^3 + mt + m$ for a particular value of $m$ within the range specified.



If $27 + 4m > 0$, the radical in Equation (3), Plate 62, is the square root of a positive quantity and thus real. Accordingly,

$$u^3 = R,$$

where R is a real quantity constructible from the given coefficient $m$ by straightedge and compasses. The three roots of this are:

$$\sqrt[3]{R}, \qquad \omega\cdot\sqrt[3]{R}, \qquad \omega^2\cdot\sqrt[3]{R},$$

and one of the roots, $t_1$, of $t^3 + mt + m = 0$ is thus real. This root may be determined by the marked ruler construction that follows:

FIG. 1. Draw a circle with center O and radius PQ = 1, upon which the chord XZ of length R/4 is marked. Extend XZ to K so that ZK = XZ = R/4, and draw KO. Now draw XM parallel to KO and insert the marked ruler so that P falls on the line XZ while Q falls on XM with PQ extended through O. The distance PX is then the cube root of R. For,

if we let PX = x; PY = y; we have from similar triangles:

$$(PX)/(PQ) = (XK)/(QO) \qquad \text{or} \qquad x = R/2y \qquad \ldots\ldots\ldots\ldots(1)$$

(since QO = 1 + QY = PQ + QY = PY = y).

From the secant property of the circle, however:

$$(PX)(PZ) = (PY)(PW) \qquad \text{or} \qquad x(x + R/4) = y(y + 2) \qquad \ldots\ldots\ldots\ldots(2)$$

Combining (1) and (2) to eliminate y, we have:

$$x(x + R/4) = (R/2x)(R/2x + 2)$$

or

$$(x^3 - R)(4x + R) = 0,$$

one of whose solutions is $x = \sqrt[3]{R}$. What is the position of the ruler corresponding to the factor $(4x + R) = 0$?

Notice that the foregoing construction solves the problem of inserting two geometric means between the quantities 1 and R; that is, two quantities x and $x^2$ such that 1, x, $x^2$, R shall form a geometric progression.

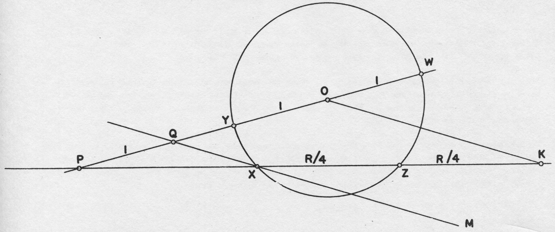
SUMMARY: We have thus established the following important theorem:

ALL GEOMETRICAL CONSTRUCTIONS WHOSE ANALYTICAL FORMULATION LEADS TO CUBIC OR QUARTIC EQUATIONS WHOSE COEFFICIENTS REPRESENT GIVEN OR CONSTRUCTED LENGTHS CAN BE SOLVED BY COMPASSES AND MARKED RULER IN THE "INSERTION" MANNER; that is, EITHER AS A PROBLEM OF TRISECTION OR AS ONE DEPENDENT UPON THE EXTRACTION OF A CERTAIN CUBE ROOT.

Determine the character of the roots of the following cubics, then give marked ruler solutions in the allotted spaces:

FIG. 2. $t^3 - 27t - 27 = 0$ (Hint: Let $t = 3x$.)

FIG. 3. $4t^3 + 21t + 21 = 0.$

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# THE CARPENTER'S SQUARE

The Carpenter's Square considered here has parallel edges. We shall assume the ability to move one corner along a fixed line while an edge of the Square passes always through a fixed point. We take the width of both portions as unity; that is, in Fig. 1, BP = PD = DE = 1.

FIG. 1. In order to trisect a given angle BOF, first construct with the Square the line DD' parallel to OF at a unit's distance from it. Then move the Square so that its corner D travels along DD' while the inner edge PO passes through O. When the other corner B falls on the second side of the given angle, this angle is trisected. Why?

FIG. 2 Newton (see Enriques and S. Roberts) used the Square under the same sliding process to draw the Cissoid of Diocles. The corner D moves along a fixed line CD while the outer edge BA passes through the fixed point A, located 2 units distant from CD. The path of the midpoint P of BD is the cissoid. Let AC be the X-axis and its perpendicular bisector be the Y-axis. Then

$$BD = AC = 2 \quad \text{and} \quad AB = DC.$$

Let $P = (x, y)$; $B = (h, k)$; $D = (1, z)$. Then, P being the midpoint of BD, we have:

$$x = (1 + h)/2, \quad y = (z + k)/2 \quad \text{or} \quad h = 2x - 1, \quad k = 2y - z. \quad \dots\dots\dots\dots(1)$$

Now in all positions AB = CD. Accordingly, $(1 + h)^2 + k^2 = z^2$, or, using (1):

$$x^2 + y^2 = yz \quad \dots\dots\dots\dots(2)$$

Since AB is perpendicular to BD their slopes are negative reciprocals. Thus

$$k/(1 + h) = (1 - h)/(k - z) \quad \text{or using (1):} \quad (2y - z)(y - z) = 2x(1 - x).$$

Substituting here the value of $z$ from (2), we have finally:

$$y^2 = x^3/(2 - x)$$

the equation of the Cissoid having $x = 2$ as <u>Asymptote</u> and <u>cusp</u> at (0, 0).

FIG. 3. The cube root of a segment R may be determined by the Carpenter's Square. Let OL = 2 and its perpendicular OM = R, OT = 2R. Draw LT and move the Square through A as in Fig. 2, until P lies on LT. Draw MNP. Then

$$LN = \sqrt[3]{R}$$

Proof: The equation of the Cissoid derived above may be rewritten in the form:

$$(y/x)^3 = y/(2 - x).$$

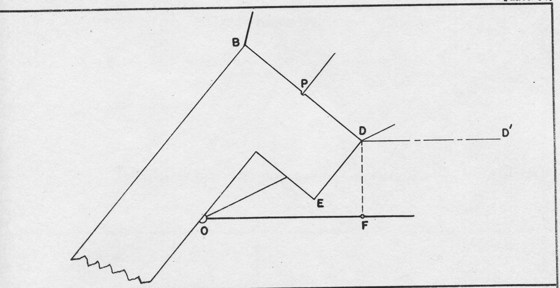
Now a line OS; $y/x = m$ through the origin O, cuts the curve in a point P whose coordinates (x, y) satisfy

$$m^3 = y/(2 - x).$$

But this equation may also be thought of as a line through (2, 0) and P; that is, the line LT. Its Y-intercept is $OT = 2m^3$. Since $LS = 2m$ and $LN = (LS)/2$, $OM = (OT)/2$, then

$$(LN)^3 = OM = R.$$

Thus: THE CARPENTER'S SQUARE USED IN THE MANNER INDICATED IS CAPABLE OF SOLVING ALL PROBLEMS OF THE FOURTH DEGREE WHOSE REPRESENTATIVE EQUATIONS HAVE POSSESSED LENGTHS AS COEFFICIENTS.

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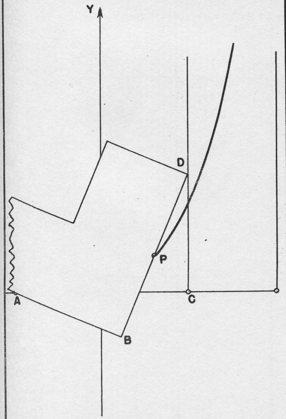


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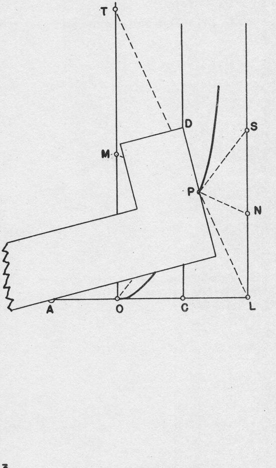


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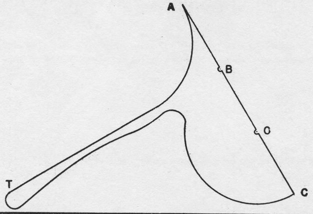
THE TOMAHAWK

FIG. 1. A semicircle upon BOC as diameter is attached to the straightedge TB such that TB is its tangent at B. A, B, O, and C are collinear with AB = BO = OC = 1.

FIG. 2. Trisect a selected angle by means of the Tomahawk. (Hint: See Plate 64,1.)

FIG. 3. Take the cube root of a selected segment by means of the Tomahawk. (Hint: See Plate 64,3.)

Is the Tomahawk capable of producing solutions of all quartic equations? ____________________.
State the privileges under which it is used.

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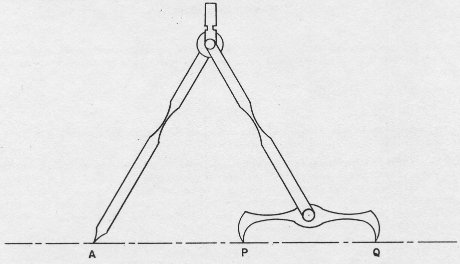
THE COMPASSES OF HERMES

FIG. 1. Consider the compasses with three feet given by H. Hermes in 1883. Here two points, P and Q, attached to one leg of the compasses at a constant distance apart, are always in line with A, the foot of the other leg.

FIG. 2. Use the Compasses of Hermes to trisect a selected angle. (See Plate 61,5.)

FIG. 3. Use the Compasses of Hermes and the straightedge to take the cube root of a selected segment. (See Plate 63,1.)

Is the Compasses of Hermes a quartic tool? _______________. Explain.

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TWO RIGHT ANGLE RULERS

We have seen that the right angle ruler is capable of accomplishing all straightedge-compasses constructions (see Plate 58). All quartics with given lengths as coefficients can be reduced to cubics of the sort

$$x^3 - px - q = 0$$

by rational transformations and this reduction may be effected by a single right angle ruler.

FIG. 1. We are able to solve cubics with two right angle rulers if we assume the ability to move the vertices of the rulers along selected lines. Upon the two perpendicular lines X, Y, lay off AO = 1, OB = p, BC = q. Place one edge of one ruler through A, an edge of the other through C so that their two other edges are together. If they are adjusted so that the vertex of the first ruler lies upon the line X, that of the second upon Y, then OM = x is a root of the given cubic. For, since the right triangles OAM, OMN, and BNC are similar:

$$x = (p + z)/x = q/z \qquad \text{(where } z = BN\text{)},$$

or eliminating z:

$$x^3 - px - q = 0.$$

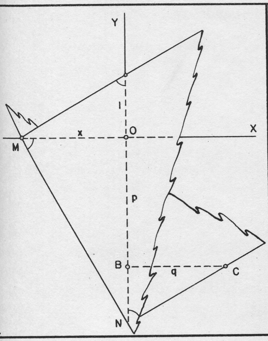
FIG. 2. The extraction of a cube root is obtained by taking AO = 1, OC = a (the foregoing situation for p = 0) and adjusting the rulers as shown. It is not difficult to see that

$$x = \sqrt[3]{a}.$$

FIG. 3. With two right angle rulers, trisect 60°.

FIG. 4. With two right angle rulers, duplicate a cube.

ALL GEOMETRICAL CONSTRUCTIONS WHOSE ANALYTIC FORMULATION LEADS TO QUARTIC EQUATIONS WHOSE COEFFICIENTS REPRESENT POSSESSED LENGTHS MAY BE SOLVED WITH TWO RIGHT ANGLE RULERS IN THE "SLIDING" MANNER.

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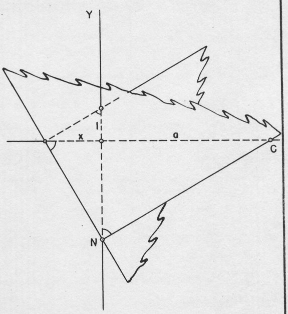


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## THE STRAIGHTEDGE AND COMPASSES WITH IMMOVABLE CONIC

In accordance with the elementary geometry of Poncelet-Steiner, we shall assume a fixed conic located somewhere in the plane and a movable straightedge or movable compasses.

FIG. 1. Let the conic be represented by the equation:

$$y^2 + ax^2 + bx + c = 0.$$

where a, b, c, are given unalterable constants. The movable straightedge puts us in possession of all lines

$$y = mx + p,$$

where m and p are at our disposal. The X-coordinates of the intersections of such lines with the conic are, eliminating y between the two equations:

$$(m^2 + a)x^2 + (2mp + b)x + (p^2 + c) = 0.$$

Evidently, by the selection of the quantities m and p, this quadratic may be made to represent all quadratics. Accordingly,

THE STRAIGHTEDGE AND FIXED CONIC WILL SOLVE ALL CONSTRUCTIONS OF A QUADRATIC NATURE.

FIG. 2. Given a fixed conic and a variable compasses. As the fixed conic, we take the parabola:

$$y = x^2$$

The compasses gives all circles: $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$, with centers (h, k) and radii r. The parabola meets the variable circle in points whose abscissas are given by

$$x^4 + (1 - 2k)x^2 - 2hx + h^2 + k^2 - r^2 = 0. \qquad \text{...........(1)}$$

These coefficients may take on all values and since every quartic with constructible coefficients is reducible to one of this type with constructible coefficients, then

THE COMPASSES AND FIXED PARABOLA (OR CONIC) WILL SOLVE ALL CONSTRUCTIONS OF A QUARTIC NATURE.

FIG. 3. To illustrate, trisect $60^\circ$ with the compasses and fixed parabola. The trisection Equation for $60^\circ$ is (see Plate 61,3):

$$x^3 - 3x - 1 = 0.$$

Equation (1) above will reduce to this if

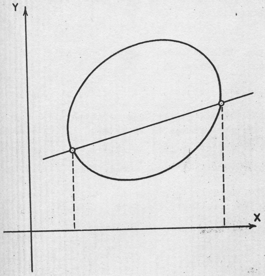
$$r^2 = h^2 + k^2, \qquad h = 1/2, \qquad k = 2;$$

that is, if the circle passes through the origin with center at (1/2, 2). The values x satisfying the trisection equation are the abscissas of the points of intersection of this circle and the given parabola. Make the drawing of the parabola and the circle showing the angle $60^\circ$ and its trisection. (See Plate 61.) In how many points does the circle cut the parabola? Explain.

FIG. 4. Duplication of the Cube: Construct the length x such that $x^3 = 2$. (Hint: In Equation 1, select

$$r^2 = h^2 + k^2, \qquad h = 1, \qquad k = 1/2.)$$

Make the construction. In how many points does the circle cut the parabola? Explain.

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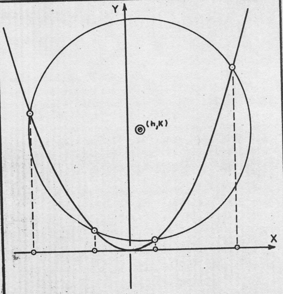


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## SECTION XI

## GENERAL PLANE LINKAGES

This section introduces the subject of general linkage motion. It is to be understood that time and space do not permit elaboration and further study from the given references must be made in order to catch something of the breadth and spirit of the subject.

The simplest linkage - the Three Bar mechanism - is especially interesting. The curves generated by the various forms of this linkage offer a challenging analysis that has attracted many of the best mathematical minds. A thorough knowledge of this linkage very often presents the key to understanting more involved mechanisms.

Although the subject matter has been investigated exhaustively, there still remain some unanswered questions. Two of these are the following:

1. What simple linkages will serve to transform the circle into an airfoil? (The mechanism must, of course, be practical.)

2. What linkage will describe the conic through five named points? (An answer to this may well come through the theorem of Pascal.)

In making models of the various linkages, the student should obtain colored cardboard about 12-ply; an eyelet punch; and boxes of #2 and #3 eyelets. Use the #2 eyelet to join two links; #3 to join three or four links. Cut the cardboard into strips about one-half inch wide with a phototrimmer and mount the model on a cardboard background. To insure greater accuracy, two bars of the same length should be punched simultaneously.

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FIG. 1. The 3-bar linkage shown was devised by James Watt, of steam engine fame, about 1784. The midpoint P of the traversing bar describes an approximately straight line. In some letters Watt said: "...about 5 feet in the height of the (engine) house may be saved in 8 feet strokes which I look upon as a capital saving;..." and "...though I am not over anxious after fame, yet I am more proud of the parallel motion than of any other invention I have ever made."

Show that if

$$AB = 2a; \qquad CP = PD = a; \qquad AC = BD = a\sqrt{2},$$

the path of P is the Lemniscate.

FIG. 2. This mechanism, devised by Tschebyscheff about 1850, is a better line approximation than the one of Watt. Here $AB = 4a$; $DP = PC = a$; $AC = BD = 5a$ and P traces the approximately straight line.

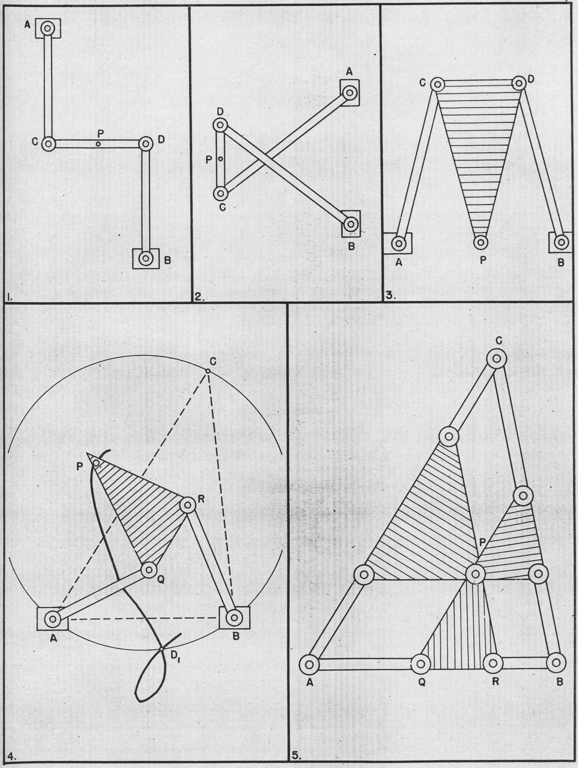
FIG. 3. A still better approximation to line motion is that path of P, attached to the plate shown, where $AC = PC = PD = DB$ and $AB = 2(CD)$. This was devised by R. Roberts about 1860.

FIG. 4. The general 3-bar mechanism produces a complicated curve of the sixth degree. If the triangle ABC be formed similar to the plate PQR, the circumcircle of ABC will pass through the double points of this sextic curve. (See Morley, F. V. Read this article and append some notes here.)

FIG. 5. This exhibits a most remarkable property of the 3-bar linkage. Select a triangle ABC and any internal point P. Draw lines through P parallel to the sides of ABC, thus determining a triple 3-bar mechanism as shown. (The 3-bar part ABPQR of Fig. 5 might, for example, be the same as that in Fig. 4 when extended.)

Now, no matter how the linkwork be deformed, triangle ABC remains always similar to itself. That is, for instance, if A and B are fixed, P describes a 3-bar curve, and the free point C remains at rest. Or, if A, B, and C are fixed to the plane, all three of the 3-bar mechanisms produce the same curve in mutual harmony and cooperation.

Plate 69.



A SPECIAL THREE-BAR LINKAGE

FIG. 1. Consider the trapezoid, ABCD of Fig. 1. Let altitudes h from B and C be dropped to the points M and N. Obviously, BC = MN, AM = ND = u. Let AC = BD = 2b; AB = CD = 2a, where a > b. Then from the figure:

$$h^2 + u^2 = 4b^2; \qquad (AD - u)^2 + h^2 = 4a^2.$$

Subtracting these:

$$(AD)^2 - 2u(AD) = 4(a^2 - b^2).$$

or

$$(AD)(AD - 2u) = (AD)(BC) = 4(a^2 - b^2).$$

FIG. 2. The Hart crossed parallelogram shown here with one bar AB attached to the plane, is a trapezoid. As it moves, the product of the variable distances AD and BC, according to the preceding paragraph, remains constant and equal to the difference of the squares of the lengths of radial arm and traversing bar.

We select a fixed point P on the traversing bar and draw the line OP parallel to AD and BC. It is clear that OP remains parallel to these lines and O is thus a fixed point of the line AB.

Let OP = r; OM = c, where M is the midpoint of AB; and angle POB = θ. Then from the figure:

$$r = 2(c + z)\cos\theta$$

$$BC = 2(BT)\cos\theta = 2(a - z)\cos\theta$$
$$AD = 2(AT)\cos\theta = 2(a + z)\cos\theta.$$

From the last two equations:

$$(BC)(AD) = 4(a^2 - z^2)\cos^2\theta = 4(a^2 - b^2).$$

Combining this result with the first equation to eliminate z, we have:

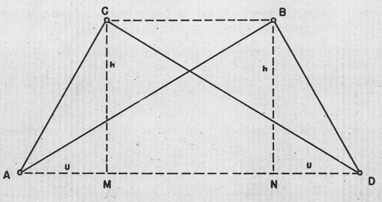
$$a^2\cos^2\theta - (r/2 - c.\cos\theta)^2 = a^2 - b^2.$$

This is the polar equation of the path of P. The quantity <u>c</u> is determined, of course, as soon as the point P is selected.

Taking a > b, select three points P on your apparatus and describe the curves:

1. When b > c,
2. When b < c,
3. When b = c.

Give the polar equation of the curve and identify when c = 0 and a = b/2.



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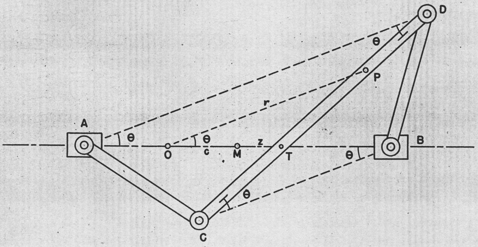

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The three-bar curve of Plate 70, traced out by a point P on the traversing bar is

$$a^2\cos^2\theta - (r/2 - c.\cos\theta)^2 = a^2 - b^2,$$

where 2a and 2b are the lengths of traversing bar (CD), and radial bar (AC = BD), respectively, and c is the distance of the tracing point from the center of this bar (CD).

If we invert this curve, taking O as the center of inversion, so that the transformation is

$$r.s = 2k^2,$$

we obtain

$$a^2s^2\cos^2\theta - (k^2 - c.s.\cos\theta)^2 = s^2(a^2 - b^2).$$

This inverted curve is a conic section which may more easily be recognized by transferring to rectangular coordinates, using

$$s.\cos\theta = x, \qquad s.\sin\theta = y, \qquad s^2 = x^2 + y^2.$$

Thus, we have:

$$(c^2 - b^2)x^2 + (a^2 - b^2)y^2 - 2c.k^2x + k^4 = 0.$$

Now, since $a > b$, the coefficient of $y^2$ is positive and the character of the conic is determined entirely by the coefficient of $x^2$. Thus the curve is

I. a Parabola if $c = b$
II. an Ellipse if $c > b$
III. an Hyperbola if $c < b$.

In all three of the accompanying figures, we have arbitrarily taken $a = 2b$. The point P' traces the conic.

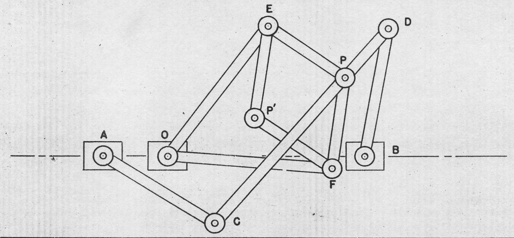
FIG. 1. shows the linkage for a parabola with $a = 2b = 2c$. Thus, $PD = AO = b$. The point P is inverted to P' by means of the Peaucellier cell where $(OE)^2 - (PE)^2 = 2k^2$. Give rectangular and polar equations.

FIG. 2. is the arrangement for an ellipse, where $2a = 4b = 3c$. For the sake of variety, P is inverted to P' by the Hart cell EFGH. Give rectangular and polar equations.

FIG. 3. gives the arrangement for an hyperbola where $a = 2b$, $c = 0$. (P is the midpoint of CD.) Give rectangular and polar equations.

Discuss the linkage in which $a = b$. Consider $c = 0$; $c \neq 0$.

Discuss the linkage if $a = c > b$.

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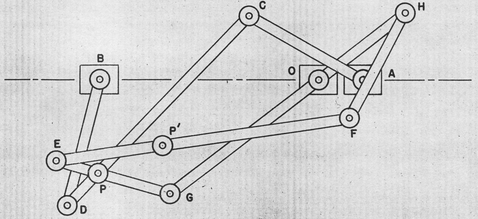


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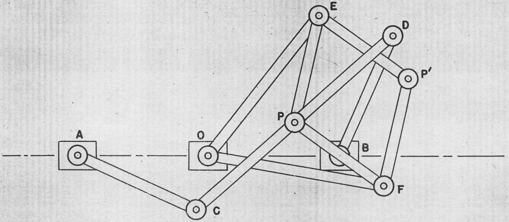



3.

PARALLELOGRAMS

FIG. 1. Consider the rhombus AMP'N with two adjacent legs extended to points O and P so that O, P', P are collinear and OM = AM. Then triangles OMP' and OAP are always similar and thus

$$OM/OA = 1/2 = OP'/OP, \quad \text{or} \quad OP = 2(OP').$$

Accordingly, if O be fixed and P be moved on some curve, the point P' traces a curve similar and similarly placed to the first and reduced in size by 1/2. This is the form of the ordinary Pantograph.

FIG. 2. This is an obvious extension of the Pantograph with multiple tracing points. What are the reduction factors for P'?________________, P"?________________, P"'?________________.

FIG. 3. Referring to Fig. 1, the bar NP' may be extended to meet an additional bar OB without affecting the character of the linkage. Thus the bar MP' may be discarded as in Fig. 3 to give the Pantograph built upon the general parallelogram OBA, with O, P', and P collinear.

FIG. 4. Five rhombuses are jointed together as shown. In all positions, M is the midpoint of BC, while G is the lower trisecting point of AM. Thus G is always the centroid of the variable triangle ABC.

FIG. 5. The linkage shown is the crossed parallelogram OABC with a short side, OC, fixed to the plane. As the mechanism moves, the bars OA and BC slide over each other and their point of intersection P describes an ellipse with O and C as foci. For,

$$OP + PC = OP + PA = OA = \text{constant}.$$

But, for like reasons, P lies always on an ellipse of the same size having A and B as foci. This second ellipse touches the fixed ellipse at P and the motion is that of one ellipse rolling upon another.

What is the path of a focus of an ellipse that rolls upon a fixed ellipse of the same size?

FIG. 6. Here one of the longer bars, OA, of the crossed parallelogram is fixed to the plane. Show that the lines OC and AB extended meet on an hyperbola with O and A as foci.

Notice that the two positions in which OC and AB are parallel define the directions of the asymptotes. The motion here is that of one hyperbola rolling upon another. Sketch them in for the position of the linkage shown.

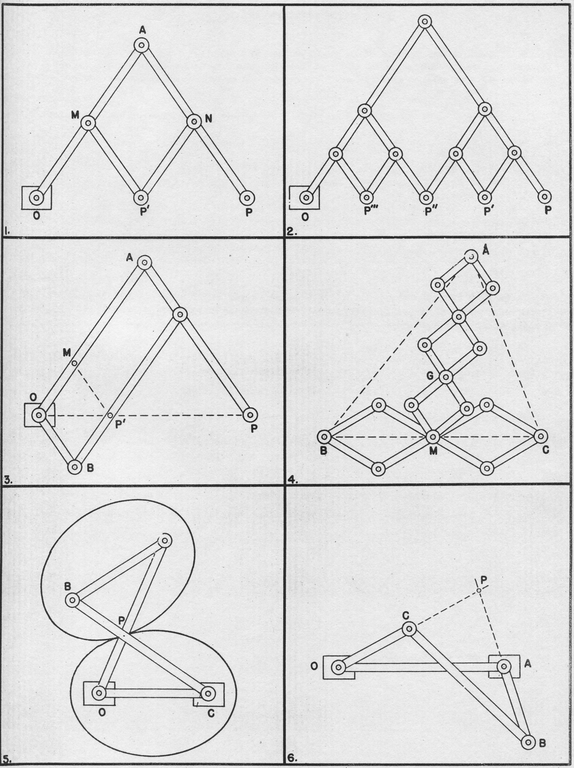

A
M
N
O
P'
P
1.
O
P'''
P''
P'
P
2.
A
M
O
P'
P
B
3.
A
G
B
M
C
4.
B
P
O
C
5.
P
C
O
A
B
6.

FIG. 1. The linkage shown is formed of two parallelograms. If O and O' are fixed to the plane so that the horizontal and vertical projections of OO' are h, k, the point P may be moved (within the limits of the mechanism) to any position in the plane. The position of the point P' is determined by P. It is clear from the figure that:

$$x' = x + h \qquad\qquad y' = y + k$$

the relation between P and P'. This is <u>simple translation</u> that is met in the study of analytic geometry.

FIG. 2. Consider the parallelogram, two of whose adjacent legs are replaced by <u>positively similar plates.</u> Let r be the ratio of the lengths of the sides of the plates which form the angle θ and let the point O be a fixed origin of the complex number system. Denote the ends of the bars by the complex variables <u>a</u>, <u>b</u>, and the unjointed vertices of the plates by $W:(x' + iy')$, $Z:(x + iy)$. Then, since the plates are similar:

$$W - a = Kb, \qquad a = K(Z - b) \qquad \text{where } K = re^{i\theta}$$

Accordingly,

$$W = KZ.$$

Thus the length OW is a constant multiple (r) of the length OZ while the angle WOZ is always equal to θ. In other words, triangle WOZ <u>remains always similar</u> to the triangular plates. The mechanism is the <u>Skew Pantograph</u> of Sylvester.

If $r = 1$, the foregoing relation is

$$W = e^{i\theta}Z$$

or

$$x' + iy' = (\cos\theta + i\sin\theta)(x + iy).$$

Equating reals and imaginaries:

$$x' = x\cos\theta - y\sin\theta$$

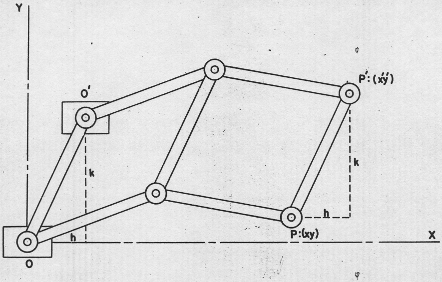
$$y' = x\sin\theta + y\cos\theta$$

the <u>equations of rotation</u> which play an important role in Analytics.

Combine the linkages of Figs. 1 and 2 to obtain a mechanism for simultaneous translation and rotation.

What is the nature of the triangular plates when $r = 1$?

What effect is there on the relation $W = KZ$ if the lengths <u>a</u> and <u>b</u> are altered?

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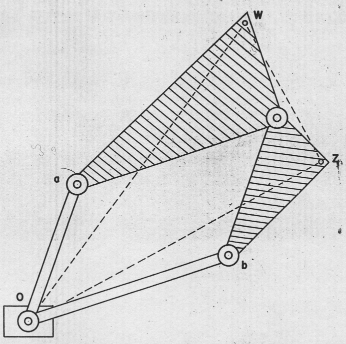



2.

FIG. 1. In the figure, let $OK = KR = b$, $OM = MP = MR = a$, and let O and K be fixed to the plane. We wish to find the path traced out by the point P at the extremity of the bar RP.

Since the points P, O, and R are always equidistant from M, they lie on the circle with center at M. Accordingly, POR is always a right angle. Using a system of polar coordinates with O as pole and OK as polar axis, let $OP = s$. Then:

$$s^2 = (2a)^2 - (OR)^2.$$

But $$OR = 2b.\cos(90^\circ - \theta) = 2b.\sin\theta.$$

Thus $$s^2 = 4a^2 - 4b^2\sin^2\theta,$$

the polar equation of the path of P.

FIG. 2. Let us invert this curve by means of a Peaucellier cell whose fundamental relation is:

$$r.s = 2k^2.$$

We shall have

$$k^4/r^2 = a^2 - b^2\sin^2\theta,$$

or $$a^2r^2 - b^2r^2\sin^2\theta = k^4,$$

as the polar equation of the inverted curve. Transferring to rectangular coordinates with $r.\cos\theta = x$, $r.\sin\theta = y$:

$$a^2x^2 + (a^2 - b^2)y^2 = k^4,$$

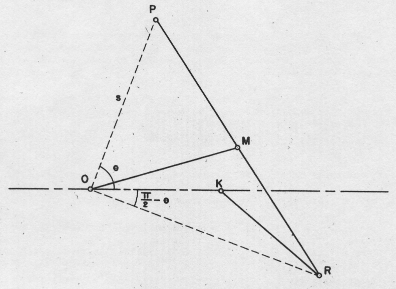
the equation of the path of P' in Fig. 2. This is a <u>central conic</u> whose character is determined by the sign of the coefficient of $y^2$. Thus

an Ellipse if $a > b$

an Hyperbola if $a < b$.

Discuss the linkage and the paths of P and P' if $a = b$.

Give in terms of <u>a</u> and <u>b</u> the coordinates of the foci of the conics.

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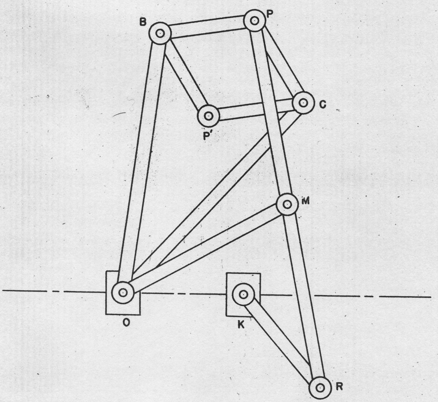



2.

# THE LIMACONS
(Roulettes)

FIG. 1. The members of the Limacon Family may be generated in the manner of the Epicycloids - that is, from one circle rolling upon another without slipping. Upon the fixed circle of radius $a$, rolls another of the same radius. Any point P, rigidly attached to the moving circle at a distance $b$ from its center, generates a Limacon.

Let the original position of B' be B. Then arc BT = arc B'T, where T is the point of tangency, and accordingly angle ACB = angle CAB' = $\theta$. Take the origin of coordinates at O, a distance $b$ from C on CB. Dropping perpendiculars from O and P upon AC, it is clear that

$$r = 2a - 2b.\cos\theta$$

is the polar equation of the path of P. The three types of this family are defined when

$b < a$ (P interior to the rolling circle)

$b = a$ (P on the rolling circle) (The Cardioid)

$b > a$ (P exterior and attached to an extension of a diameter)

Sketch a Limacon of each type on the given diagram.

FIG. 2. Two similar (proportional) crossed parallelograms are attached as shown with

$$DB' = EA = a; \qquad FD = CE = c; \qquad AB' = DE = FC = \sqrt{(ac)}.$$

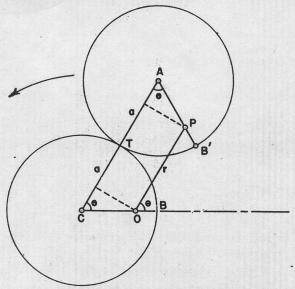
Then (see Plate 38.4), angles FDE = EDB' = FCE = CAB' = $\theta$. Accordingly, if F and C are fixed, while AC moves through an angle $\theta$ about C, the bar AB' swings about A through the same angle. This is the action of the rolling circles explained in Fig. 1 and thus any point P of AB' describes a Limacon.

Draw the circles to fit the mechanism and locate the point P that describes the Cardioid.

FIG. 3. A very similar linkage is given by Hebbert. Again two similar crossed parallelograms FCDK and CGED, are taken to produce equal angles $\theta$ at the fixed point C.

Upon the bars CG and CD is erected a parallelogram CHJA of arbitrary size, one of whose sides, JA, is extended to P. Then angle HJA = angle CAP = $\theta$. This produces the same action as displayed in the two preceding figures and thus P describes the Limacon.

Draw two circles to fit the mechanism.

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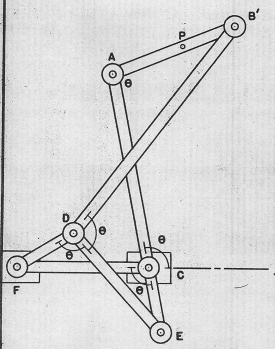


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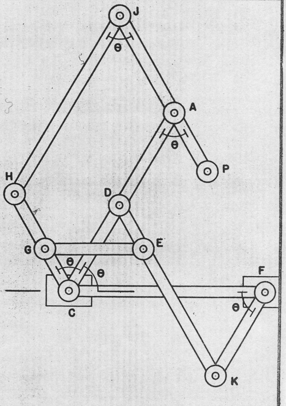



3.

# CISSOIDS

FIG. 1. Consider the arrangement of the Peaucellier cell shown. The points D and Q are fixed to the plane so that $DO = DQ = c$; $OA = OB = a$; $AQ = QB = BP = PA = b$. We take the line DQ as axis, Q as pole, and find the polar equation of the path of $P:(r, \theta)$. From the fundamental property of the cell:

$$(OQ)(OQ + r) = a^2 - b^2, \qquad (a > b).$$

But, since ODQ is an isosceles triangle, $OQ = 2c.\cos\theta$. Accordingly,

$$r = (a^2 - b^2)/2c.\cos\theta - 2c.\cos\theta, \quad \text{or} \quad y^2 = x^2(4c^2 - a^2 + b^2 + 2cx)/(a^2 - b^2 - 2cx). \qquad \text{...........(1)}$$

These are members of the Cissoid family. What are their inverses with respect to the origin?

FIG. 2. The same curves may be generated with a fifth bar attached to the Hart cell as shown. The points Q and D are fixed, O travels on a circle through Q, while P traces the curve. (O, Q, P are collinear and $OQ.OP = k^2$.)

FIGS. 3, 4, 5. Sketch the three members of the family of Cissoids, Equations (1), for the relative values indicated. (Take, for instance, $a = 5$, $b = 3$, $c = 3, 2, 1$.) What is the nature of the curve at the origin in each instance?

FIG. 6. We have already shown that the Cissoid may be used to extract the cube root of a segment R. (See Plate 64,3.)

To trisect a given angle AOB, proceed as follows: Draw the unit circle meeting the sides of the angle in A, B and establish its cosine: $a = OR$. Upon a line through O perpendicular to OB, let $OT = 1/a$. Let $OM = (OB)/2$. Through the point P, where the line MT meets the Cissoid, draw OP produced to meet the line $y = 1/2$ in Y. Drop the perpendicular from T to OB meeting the circle in X. Then

$$\text{arc BX} = (\text{arc BA})/3.$$

For,

if OB and OT are coordinate axes, the path of P: $y^2 = x^3/(2 - x)$ meets the line OY: $ry = x$ in a point whose ordinate is:

$$y = 2/r(1 + r^2).$$

The line $ry = x$ meets MT: $ay = 1 - 2x$ in

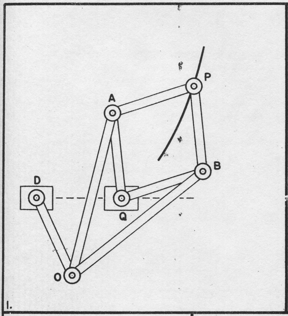
$$y = 1/(a + 2r).$$

If these points fall together at P, then $2/r(1 + r^2) = 1/(a + 2r)$, or

$$r^3 - 3r - 2a = 0.$$

This is the Trisection Equation where $r = 2\cos(AOB/3)$. But $r = \cot(YOB) = 2(SY)$. Then, since $\cos(XOB) = SY$, the statement is evident.

THUS, WITH THE COOPERATION OF THE STRAIGHTEDGE AND BY PROPER FASTENING AND SELECTION OF LENGTHS, EITHER MECHANISM IS A QUARTIC TOOL.



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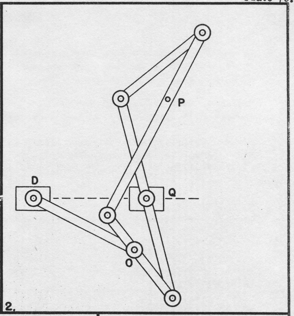

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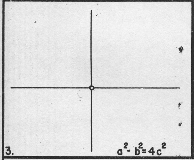

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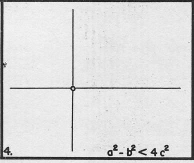

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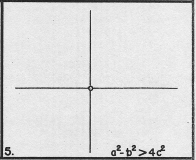

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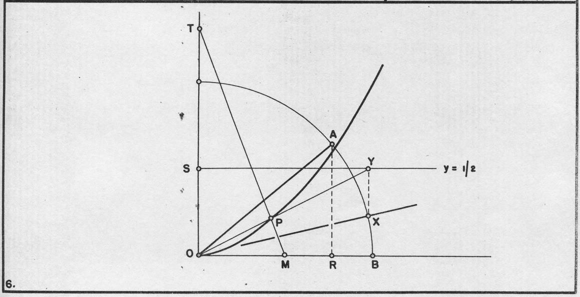


6.

# OVALS OF CASSINI

It is well known that if a point moves in a plane so that the sum, difference, quotient of its distances to two fixed points in the plane is constant, the locus generated is, respectively, an ellipse, hyperbola, circle.

FIG. 1. The Ovals of Cassini are defined as the locus of a point P moving so that the product of its distances to two fixed points A, A' (at a distance a apart) is constant ($= c^2$).

Take the midpoint O of AA' as origin and AA' as axis. Find the polar and rectangular equations of the Ovals. Identify the curve for $a = 2c$.

Sketch in colors the locus for each of the conditions: (1): $a > 2c$; (2): $a = 2c$; (3): $a < 2c$.

FIG. 2. The linkage shown has $AB = AO = OA' = a/2$; $BC = CO = CQ = QD = OD = b/2$ with A and O attached to the plane.

Take AOA' as axis and let the coordinates of Q and P be $(s, \theta)$ and $(r, \theta)$, respectively. Since $BC = CQ = OC = b/2$, the points B, O, and Q lie on a circle with center at C. Thus the lines BO and OQ are always at right angles and $\angle BOA = \angle OBA = 90^\circ - \theta$. Then, from the right triangle BOQ:

$$(OQ)^2 = (BQ)^2 - (OB)^2,$$

or

$$s^2 = b^2 - a^2\sin^2\theta.$$

Now the equation of the path of P is obtained from this last relation through the fundamental property of the Peaucellier cell:

$$r(r - s) = k^2, \qquad \text{where} \qquad k^2 = (CP)^2 - (CQ)^2.$$

That is, eliminating s:

$$(r^2 - k^2)^2 = b^2r^2 - a^2r^2\sin^2\theta,$$

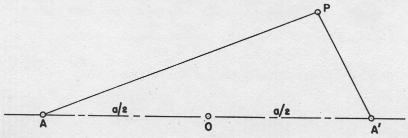
or in rectangular coordinates:

$$(x^2 + y^2)^2 - (2k^2 + b^2)x^2 + (a^2 - b^2 - 2k^2)y^2 + k^4 = 0$$

which can be identified as the Ovals of Cassini, with the fixed points (foci), A, A'.

Find relative values of a, b, k which will produce the Lemniscate. For these values what happens to the Peaucellier cell?

What is the path of P'?

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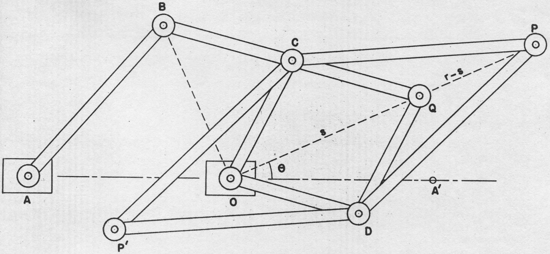



2.

# PEDALS

The PEDAL of a curve $f(x, y) = 0$ with respect to a fixed point P is the locus of the intersection of a tangent to the curve and its perpendicular from P as the tangent moves around the curve.

FIG. 1. The tangent to the Parabola $y^2 = 4ax$, for all values of $\underline{m}$, is $y = mx + a/m$. Its perpendiculars from the vertex, V, and focus, F, are respectively $my + x = 0$ and $my + x = a$. Eliminate m to find

(1) The pedal of the Parabola with respect to its focus;

(2) The pedal of the Parabola with respect to its vertex.

What is the asymptote of the pedal of (2)? Sketch both pedals in colors.

FIG. 2. The Pedal of the Ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with respect to a focus: $[\sqrt{(a^2 - b^2)}, 0]$ is the locus of intersections of the tangent $y = mx + \sqrt{(m^2a^2 + b^2)}$ and the perpendicular from F: $my + x = \sqrt{(a^2 - b^2)}$. Eliminate m to find its equation, sketch and identify.

FIG. 3. The pedal of the Rectangular Hyperbola $x^2 - y^2 = a^2$ with respect to its center is found from the tangent: $y = mx + a\sqrt{(m^2 - 1)}$ and the perpendicular line: $my + x = 0$. Obtain the equation of this pedal in both rectangular and polar coordinates, sketch and identify.

FIG. 4. The line: $kx + y\sqrt{(1 - k^2)} = a$, for all values of k, is tangent to the circle: $x^2 + y^2 = a^2$. Find the pedal with respect to the points:

A: $(a/2, 0)$

B: $(a, 0)$

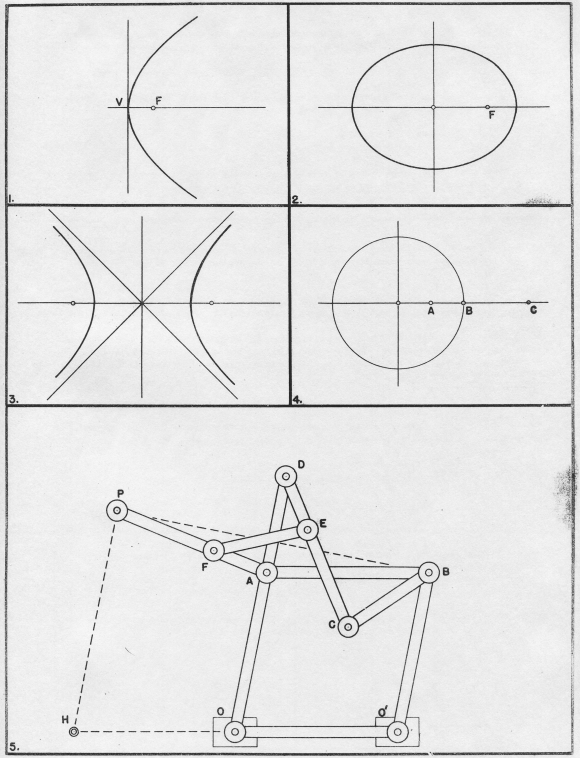
C: $(2a, 0)$

FIG. 5. Two <u>similar</u> proportional crossed parallelograms, ABCD and ADEF, are joined to produce equal angles $\theta$ at A. The bar AF is extended to P so that AP = AB; DA is extended to O so that OA = AB. Two other bars, OO' and O'B, are added to form the parallelogram shown.

Since AD bisects angle PAB, it is perpendicular to the line PB and evidently PB is always tangent to the circle described by B. The point H is taken collinear with O and O' so that HO = OO'. Then HP is parallel to OD and O'B and it is therefore perpendicular to PB at P. <u>The path of P is then the pedal of the circle O'(B) with respect to H,</u> a curve identified above as a Limacon.

Show that OD trisects the arbitrary angle POO'. (See R. C. Yates, Nat. Math. Mag., XII, 1938, pp. 323-324.)

Plate 78.



# GENERAL THEOREM OF LINKAGES

A. B. Kempe (on a General Method of Describing Plane Curves of the $n^{th}$ degree by Linkwork, Proc. Lon. Math. Soc., VII, 1876, pp. 213-216) has given the following proof that any algebraic curve may be described by a linkage.

Consider the algebraic curve: $f(x, y) = 0$. ............(1)
The parallelogram of Fig. 1 has sides $\underline{m}$ and $\underline{n}$ which make angles $\theta$ and $\phi$ with the X-axis. The vertex P is a point of the curve. Its coordinates are then:

$$x = m.\cos\theta + n.\cos\phi$$
$$y = m.\sin\theta + m.\sin\phi = m.\cos(\pi/2 - \theta) + n.\cos(\pi/2 - \phi). \qquad \text{............(2)}$$

Now the sine of any angle can be written as the cosine of its complement. Furthermore, the products and powers of cosines can be expressed as the sum of cosines. Thus, if we substitute equations (2) in (1), we shall have a sum of terms of the sort:

$$f(x, y) = \Sigma[A.\cos(a\phi \pm b\theta \pm \beta)] + C = 0, \qquad \text{............(3)}$$

where A and C are constants, $\underline{a}$ and $\underline{b}$ are positive integers, and $\beta$ equals $\pi/2$ or 0. (If $\underline{a}$ and $\underline{b}$ are rational fractions, a common denominator may be found and the function changed to integral multiples of $\phi$ and $\theta$.)

FIG. 2. The Multiplicator shown is composed of similar crossed parallelograms, discussed in previous plates. By means of mechanisms such as this we may obtain integral multiples of any angle; e.g., $a\phi$ or $b\theta$.

FIG. 3. Joining one multiplicator to another will produce the combination $a\phi \pm b\theta$. This is the mechanism shown where the plate BOK with angle $\beta$ is connected rigidly to the end bar. Thus we build up a linkage to produce $\angle$ BOX $= a\phi \pm b\theta \pm \beta$. If, in Fig. 3, OB is taken equal to A (equation 3), then the x-coordinate of the point B is

$$A.\cos(a\phi \pm b\theta \pm \beta).$$

FIG. 4. The Translator shown is composed of parallelograms with OB pivoted at O. Within the limits of the mechanism, the bar O'B' can be moved freely in the plane, remaining always parallel to OB.

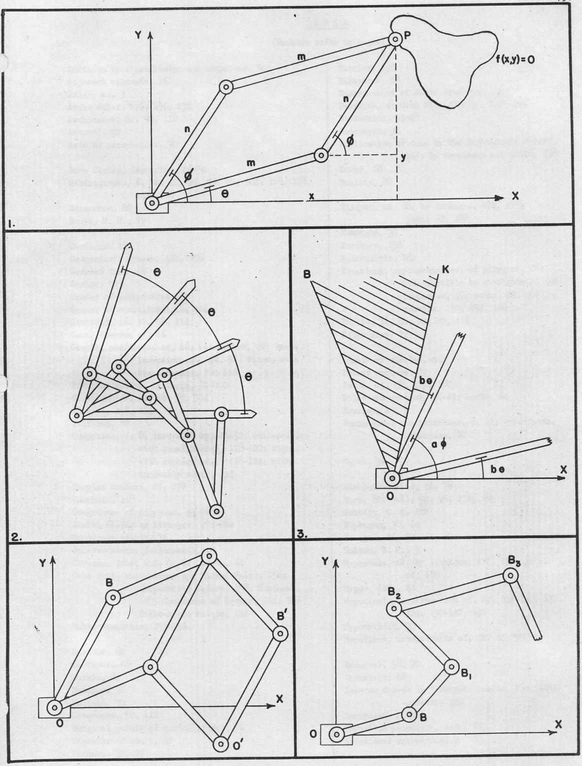
FIG. 5. By combining the linkages of Figures 1, 2, 3, and 4, we may erect a chain of links OB, $BB_1$, $B_1B_2$, . . ., as shown, whose end point, $B_n$, has x-coordinate:

$$X = \Sigma A.\cos(a\phi \pm b\theta \pm \beta)$$
$$= f(x, y) - C \text{ (by virtue of equation 3)} \qquad \text{............(4)}$$

But if P is moved along the given curve, then its coordinates x, y satisfy: $f(x, y) = 0$. Accordingly, the locus of the end point, $B_n$, of the chain is

$$X + C = 0,$$

a straight line parallel to the Y-axis. Conversely, if $B_n$ is moved along this line (with the help of a Peaucellier cell, for instance) the point P will generate the curve $f(x, y) = 0$.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